Values of the Double Sine Function

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Abstract. We calculate basic values of the double sine function. The algebraicity for some special values is proved. Its behavior in the fundamental domain is also studied. Especially we show that it has just two extremes: one relative maximum and one relative minimum. Asymptotic formulas for these extremal values are proved.

Key words: Double sine function; double Hurwitz zeta function; extremal value; algebraicity
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1 Introduction

The double sine function was discovered by Hölder [H] in 1886 as a generalization of the usual sine function. It was defined by the infinite product

$$F(x) = e^x \prod_{n=1}^{\infty} \left(\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^n e^{2x} \right).$$

Hölder proved basic properties of $F(x)$ containing the periodicity and the multiplication formula. About ninety years later, Shintani [S] constructed the normalized double sine function $F(x,(\omega_1,\omega_2))$ to investigate Kronecker's Jugendtraum for real quadratic fields. Hölder's double sine function $F(x)$ is expressed by Shintani's double sine function as $F(x) = F(1-x,(1,1))$. We studied these double sine functions in previous papers [K1]-[K4] [KK1]-[KK5] [KW1] [KW2] with various applications and generalizations. There also exists a physical application such as Jimbo-Miwa [JM]. Thus, the double sine function is an important mathematical object, but its study is still in a primitive stage. For example we do not know the precise behaviors concerning extermal values. In this paper we investigate this basic problem. The difficulty of this problem is coming from the fact that we do not have sufficient knowledge about the derivatives of the double sine function in general. We refer to [KK1] [KW1] [K3] for related results.

For later use, we recall the general construction of normalized multiple sine functions following [K1] [K2] [KK1]. The normalized multiple sine function of period $\boldsymbol{\omega} = (\omega_1, ..., \omega_r)$ for $\omega_1, ..., \omega_r > 0$ is defined as

$$S_r(x, \boldsymbol{\omega}) = \left(\prod_{n_1,\ldots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x) \right) \times \left(\prod_{m_1,\ldots,m_r=1}^{\infty} (m_1\omega_1 + \cdots + m_r\omega_r - x) \right)^{(-1)^{r-1}},$$

where the zeta reguralized product $\prod$ of Deninger [D] is used. Alternatively, $S_r(x, \boldsymbol{\omega})$ is written as

$$S_r(x, \boldsymbol{\omega}) = \Gamma_r(x, \boldsymbol{\omega})^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x, \boldsymbol{\omega})^{(-1)^r}$$

in terms of the regularized multiple gamma function

$$\begin{aligned} \Gamma_r(x, \boldsymbol{\omega}) &= \left(\prod_{n_1,\ldots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x) \right)^{-1} \\ &= \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x, \boldsymbol{\omega}) \Big|_{s=0} \right). \end{aligned}$$

Here

$$\zeta_r(s, x, \boldsymbol{\omega}) = \sum_{n_1,\ldots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

is the multiple Hurwitz zeta function defined by Barnes [B]. The case $r = 1$ is reduced to the usual gamma function and sine function:

$$\Gamma_1(x, \omega) = \frac{\Gamma(x/\omega)}{\sqrt{2\pi}} \omega^{\frac{x}{\omega} - \frac{1}{2}}$$

and

$$S_1(x, \omega) = \frac{2\pi}{\Gamma(\frac{x}{\omega})\Gamma(1 - \frac{x}{\omega})} = 2 \sin\left(\frac{\pi x}{\omega} \right).$$

For simplicity we write

$$\Gamma_r(x) = \Gamma_r(x, (1, ..., 1))$$

and

$$S_r(x) = S_r(x, (1, ..., 1)).$$

As proved in [KK1] the double sine function $F(x)$ of Hölder [H] is identified as

$$\begin{aligned} F(x) &= S_2(1-x,(1,1)) \\ &= S_2(1-x) \\ &= S_2(x)^{-1}S_1(x), \end{aligned}$$

and the double sine function $F(x,(\omega_1,\omega_2))$ of Shintani [S] is nothing but

$$F(x,(\omega_1,\omega_2)) = S_2(x,(\omega_1,\omega_2)).$$

On the other hand another type of multiple sine function $\mathcal{S}_r(x)$, which we call the *primitive* multiple sine function in [KK1], is defined as

$$\mathcal{S}_r(x) = \exp\left(\frac{x^{r-1}}{r-1}\right)\prod_{n=1}^{\infty}\left(P_r\left(\frac{x}{n}\right)P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}}$$

for $r \geq 2$ with

$$P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

Hölder's double sine function is $F(x) = \mathcal{S}_2(x)$ and $\mathcal{S}_r(x)$ is also written via $S_r(x)$ explicitly (see [KK1]).

Now we describe our results. We look at the behavior of $S_2(x,(\omega_1,\omega_2))$ on the real line. From the basic periodicity

$$S_2(x+\omega_1+\omega_2,(\omega_1,\omega_2)) = -\frac{S_2(x,(\omega_1,\omega_2))}{4\sin(\frac{\pi x}{\omega_1})\sin(\frac{\pi x}{\omega_2})}$$

proved in [KK1] in a more generalized form, we may restrict ourselves to the fundamental domain $0 \leq x < \omega_1+\omega_2$. Since $S_2(x,(\omega_2,\omega_1)) = S_2(x,(\omega_1,\omega_2))$ we may assume $0 < \omega_1 \leq \omega_2$ without the loss of generality.

First the basic values are given in the following theorem.

Theorem 1

(1)

$$S_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right) = 1.$$

(2)

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)=S_2\left(\frac{\omega_2}{2},(\omega_1,\omega_2)\right)=\sqrt{2}.$$

(3)

$$S_2\left(\omega_1+\frac{\omega_2}{2},(\omega_1,\omega_2)\right)=S_2\left(\omega_2+\frac{\omega_1}{2},(\omega_1,\omega_2)\right)=\frac{\sqrt{2}}{2}.$$

(4)

$$S_2(\omega_1,(\omega_1,\omega_2))=\sqrt{\frac{\omega_2}{\omega_1}}.$$

(5)

$$S_2(\omega_2,(\omega_1,\omega_2))=\sqrt{\frac{\omega_1}{\omega_2}}.$$

We find extremal values as follows:

Theorem 2 *Let $0<\omega_1\le\omega_2$. Then we have:*

(1) *The double sine function $S_2(x,(\omega_1,\omega_2))$ has just two extremes $S_2(\alpha_{\max}(\omega_1,\omega_2),(\omega_1,\omega_2))$ and $S_2(\alpha_{\min}(\omega_1,\omega_2),(\omega_1,\omega_2))$ in $0\le x<\omega_1+\omega_2$, where*

$$0<\alpha_{\max}(\omega_1,\omega_2)<\alpha_{\min}(\omega_1,\omega_2)<\omega_1+\omega_2.$$

(2) *$S_2(\alpha_{\max}(\omega_1,\omega_2),(\omega_1,\omega_2))$ is a relative maximum and*

$$\frac{\omega_1}{2}\le\alpha_{\max}(\omega_1,\omega_2)\le\frac{\omega_2}{2}.$$

(3) *$S_2(\alpha_{\min}(\omega_1,\omega_2),(\omega_1,\omega_2))$ is a relative minimum and*

$$\omega_1+\frac{\omega_2}{2}\le\alpha_{\min}(\omega_1,\omega_2)\le\omega_2+\frac{\omega_1}{2}.$$

(4)

$$\alpha_{\max}(\omega_1,\omega_2)+\alpha_{\min}(\omega_1,\omega_2)=\omega_1+\omega_2.$$

(5)

$$S_2(\alpha_{\max}(\omega_1,\omega_2),(\omega_1,\omega_2))S_2(\alpha_{\min}(\omega_1,\omega_2),(\omega_1,\omega_2))=1.$$

The asymptotic results for the extremal values are shown as follows:

Theorem 3

(1)

$$\lim_{t\downarrow 0}\alpha_{\max}(1,t)=\frac{1}{6}.$$

(2)

$$\lim_{t\to\infty}\frac{\alpha_{\max}(1,t)}{t}=\frac{1}{6}.$$

(3)

$$\lim_{t\downarrow 0}\alpha_{\min}(1,t)=\frac{5}{6}.$$

(4)

$$\lim_{t\to\infty}\frac{\alpha_{\min}(1,t)}{t}=\frac{5}{6}.$$

(5)

$$\lim_{\frac{\omega_1}{\omega_2}\to 0,\infty}\frac{\alpha_{\max}(\omega_1,\omega_2)}{\omega_1+\omega_2}=\frac{1}{6}.$$

(6)

$$\lim_{\frac{\omega_1}{\omega_2}\to 0,\infty}\frac{\alpha_{\min}(\omega_1,\omega_2)}{\omega_1+\omega_2}=\frac{5}{6}.$$

Theorem 4

(1)

$$\limsup_{t\downarrow 0}\frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{1/t}\le\frac{1}{6}.$$

(2)

$$\liminf_{t\downarrow 0}\frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{1/t}\ge\frac{1}{6}\log\left(\frac{3e}{\pi}\right).$$

(3)

$$\limsup_{t\to\infty}\frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{t}\le\frac{1}{6}.$$

(4)

$$\liminf_{t\to\infty}\frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{t}\ge\frac{1}{6}\log\left(\frac{3e}{\pi}\right).$$

(5)

$$\liminf_{t\downarrow 0}\frac{\log S_2(\alpha_{\min}(1,t),(1,t))}{1/t}\ge-\frac{1}{6}.$$

(6)

$$\limsup_{t\downarrow 0} \frac{\log S_2(\alpha_{\min}(1,t),(1,t))}{1/t} \leq -\frac{1}{6}\log\left(\frac{3e}{\pi}\right).$$

(7)

$$\liminf_{t\to\infty} \frac{\log S_2(\alpha_{\min}(1,t),(1,t))}{t} \geq -\frac{1}{6}.$$

(8)

$$\limsup_{t\to\infty} \frac{\log S_2(\alpha_{\min}(1,t),(1,t))}{t} \leq -\frac{1}{6}\log\left(\frac{3e}{\pi}\right).$$

We show the following algebraicity result:

Theorem 5 *Let N_1 and N_2 be positive integers with $(N_1, N_2) = 1$ or 2. Then the value $S_2(m,(N_1,N_2))$ is an algebraic number in $\mathbf{Q}\left(\sin\frac{2\pi}{N_i},\ \cos\frac{2\pi}{N_i},\ \sqrt{N_i}\ \middle|\ i=1,2\right)$ for $m = 1, ..., N_1+N_2-1$.*

Examples

$$S_2(1,(4,6)) = 1, \qquad S_2(2,(4,6)) = \sqrt{2}, \qquad S_2(3,(4,6)) = \sqrt{2},$$
$$S_2(4,(4,6)) = \sqrt{\frac{3}{2}}, \qquad S_2(5,(4,6)) = 1, \qquad S_2(6,(4,6)) = \sqrt{\frac{2}{3}},$$
$$S_2(7,(4,6)) = \frac{1}{\sqrt{2}}, \qquad S_2(8,(4,6)) = \frac{1}{\sqrt{2}}, \qquad S_2(9,(4,6)) = 1.$$

We obtain the following result in the case of a period of a special type.

Theorem 6

(1) *For an integer $n \geq 1$,*

$$S_2(n,(1,6n)) = S_2(n+1,(1,6n)) = \max\{S_2(m,(1,6n)) \mid m = 1, ..., 6n\}$$

and

$$n < \alpha_{\max}(1,6n) < n+1.$$

(2)

$$\limsup_{n\to\infty} \frac{\log S_2(n,(1,6n))}{n} \le 1.$$

(3)

$$\liminf_{n\to\infty} \frac{\log S_2(n,(1,6n))}{n} \ge \log\left(\frac{3e}{\pi}\right).$$

We make a few remarks. Algebraic natures of special values of the double sine function are very interesting from arithmetic view points. Values in Theorems 5 and 6 are examples of N-division values of the double sine function for $N = 1$ and 2. General N-division values would be important to realize Kronecker's Jugendtraum as indicated first by Shintani [S]. Moreover, extremal values appearing in Theorems 2 and 4 are quite mysterious. Our results can be generalized to the case of the multiple sine function to some extent. We will treat generalizations on another opportunity.

2 Special values: Proof of Theorem 1

Proof of Theorem 1

(1) This central value comes from the definition

$$S_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right) = \frac{\Gamma_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right)}{\Gamma_2\left(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2)\right)} = 1.$$

(2) We have

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right) = \frac{\Gamma_2\left(\frac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2)\right)}{\Gamma_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)} = \Gamma_1\left(\frac{\omega_1}{2},\omega_1\right)^{-1},$$

where we used the (quasi-)periodicity

$$\Gamma_2(x+\omega_2,(\omega_1,\omega_2)) = \Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}.$$

Notice that

$$\Gamma_1(x,\omega) = \frac{\Gamma(\frac{x}{\omega})}{\sqrt{2\pi}}\omega^{\frac{x}{\omega}-\frac{1}{2}}.$$

Hence

$$\Gamma_1\left(\frac{\omega}{2},\omega\right)=\frac{\Gamma(\frac{1}{2})}{\sqrt{2\pi}}\omega^{\frac{1}{2}-\frac{1}{2}}=\frac{1}{\sqrt{2}}.$$

Thus

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)=\sqrt{2}.$$

By symmetry we also have

$$S_2\left(\frac{\omega_2}{2},(\omega_1,\omega_2)\right)=\sqrt{2}.$$

(3) By the reflection $S_2(\omega_1+\omega_2-x,\ (\omega_1,\omega_2))=S_2(x,\ (\omega_1,\omega_2))^{-1}$ we have

$$S_2\left(\omega_1+\frac{\omega_2}{2},(\omega_1,\omega_2)\right)=S_2\left(\frac{\omega_2}{2},(\omega_1,\omega_2)\right)^{-1}=\frac{\sqrt{2}}{2}$$

and

$$S_2\left(\omega_2+\frac{\omega_1}{2},(\omega_1,\omega_2)\right)=S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)^{-1}=\frac{\sqrt{2}}{2}.$$

(4) First, we remark that

$$\begin{aligned}S_2\left(\omega_1,(\omega_1,\omega_2)\right)&=\frac{\Gamma_2(\omega_2,(\omega_1,\omega_2))}{\Gamma_2(\omega_1,(\omega_1,\omega_2))}\\&=\lim_{x\to 0}\frac{\Gamma_2(\omega_2+x,(\omega_1,\omega_2))}{\Gamma_2(\omega_1+x,(\omega_1,\omega_2))}.\end{aligned}$$

Here we use

$$\begin{aligned}\frac{\Gamma_2(\omega_2+x,(\omega_1,\omega_2))}{\Gamma_2(\omega_1+x,(\omega_1,\omega_2))}&=\frac{\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}}{\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_2)^{-1}}\\&=\frac{\Gamma_1(x,\omega_2)}{\Gamma_1(x,\omega_1)}\\&=\frac{\Gamma(\frac{x}{\omega_2})\omega_2^{\frac{x}{\omega_2}-\frac{1}{2}}}{\Gamma(\frac{x}{\omega_1})\omega_1^{\frac{x}{\omega_1}-\frac{1}{2}}}.\end{aligned}$$

Hence we obtain

$$\begin{aligned} S_2\left(\omega_1,(\omega_1,\omega_2)\right) &= \lim_{x\to 0}\frac{\Gamma(\frac{x}{\omega_2})\omega_2^{\frac{x}{\omega_2}-\frac{1}{2}}}{\Gamma(\frac{x}{\omega_1})\omega_1^{\frac{x}{\omega_1}-\frac{1}{2}}} \\ &= \lim_{x\to 0}\frac{\frac{\omega_2}{x}\omega_2^{-\frac{1}{2}}}{\frac{\omega_1}{x}\omega_1^{-\frac{1}{2}}} \\ &= \sqrt{\frac{\omega_2}{\omega_1}}. \end{aligned}$$

By symmetry (or from $S_2\left(\omega_2,(\omega_1,\omega_2)\right)=S_2\left(\omega_1,(\omega_1,\omega_2)\right)^{-1}$) we have

$$S_2\left(\omega_2,(\omega_1,\omega_2)\right)=\sqrt{\frac{\omega_1}{\omega_2}}.$$

∎

Remark From the relation

$$\begin{aligned} S_2\left(\omega_1,(\omega_1,\omega_2)\right) &= \lim_{x\to 0} S_2\left(\omega_1+x,(\omega_1,\omega_2)\right) \\ &= \lim_{x\to 0}\frac{S_2\left(x,(\omega_1,\omega_2)\right)}{S_1(x,\omega_2)} \\ &= \frac{S_2'(0,(\omega_1,\omega_2))}{S_1'(0,\omega_2)} \\ &= \frac{\omega_2}{2\pi}S_2'(0,(\omega_1,\omega_2)) \end{aligned}$$

we see that

$$\begin{aligned} S_2\left(\omega_1,(\omega_1,\omega_2)\right)=\sqrt{\frac{\omega_2}{\omega_1}} \quad &\Longleftrightarrow \quad S_2'(0,(\omega_1,\omega_2))=\frac{2\pi}{\sqrt{\omega_1\omega_2}} \\ &\underset{\text{symmetry}}{\Longleftrightarrow} \quad S_2\left(\omega_2,(\omega_1,\omega_2)\right)=\sqrt{\frac{\omega_1}{\omega_2}}. \end{aligned}$$

We refer to [K3] for a direct proof of $S_2'(0,(\omega_1,\omega_2))=\frac{2\pi}{\sqrt{\omega_1\omega_2}}$.

3 Extermal points: Proof of Theorems 2 and 3

Proof of Theorem 2

To simplify the notation we put $S(x)=S_2(x,(\omega_1,\omega_2))$ here. Then

$$S(x)=\Gamma_2(x,(\omega_1,\omega_2))^{-1}\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))$$

with

$$\Gamma_2(x,(\omega_1,\omega_2)) = \exp\left(\frac{\partial}{\partial s}\zeta_2(0,x,(\omega_1,\omega_2))\right)$$

for

$$\zeta_2(s,x,(\omega_1,\omega_2)) = \sum_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)^{-s}.$$

We notice that $\zeta_2(s,x,(\omega_1,\omega_2))$ converges absolutely in $\mathrm{Re}(s)>2$ and it has a meromorphic continuation to all $s\in\mathbf{C}$. Moreover $\zeta_2(s,x,(\omega_1,\omega_2))$ is holomorphic at $s=0$.

Since

$$\begin{aligned}\log S(x) &= -\log\Gamma_2(x,(\omega_1,\omega_2))+\log\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))\\ &= -\frac{\partial}{\partial s}\zeta_2(0,x,(\omega_1,\omega_2))+\frac{\partial}{\partial s}\zeta_2(0,\omega_1+\omega_2-x,(\omega_1,\omega_2))\end{aligned}$$

we have

$$\frac{S'}{S}(x) = -\frac{\partial^2}{\partial s\partial x}\zeta_2(0,x,(\omega_1,\omega_2))-\frac{\partial^2}{\partial s\partial x}\zeta_2(0,\omega_1+\omega_2-x,(\omega_1,\omega_2)).$$

Hence

$$\left(\frac{S'}{S}\right)'(x) = -\frac{\partial^3}{\partial s\partial x^2}\zeta_2(0,x,(\omega_1,\omega_2))+\frac{\partial^3}{\partial s\partial x^2}\zeta_2(0,\omega_1+\omega_2-x,(\omega_1,\omega_2))$$

and

$$\begin{aligned}\left(\frac{S'}{S}\right)''(x) &= -\frac{\partial^4}{\partial s\partial x^3}\zeta_2(0,x,(\omega_1,\omega_2))-\frac{\partial^4}{\partial s\partial x^3}\zeta_2(0,\omega_1+\omega_2-x,(\omega_1,\omega_2))\\ &= 2\left(\sum_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)^{-3}+\sum_{m_1,m_2\geq 1}(m_1\omega_1+m_2\omega_2-x)^{-3}\right),\end{aligned}$$

where we use the relation

$$\frac{\partial^3}{\partial x^3}\zeta_2(s,x,(\omega_1,\omega_2)) = (-s)(-s-1)(-s-2)\zeta_2(s+3,x,(\omega_1,\omega_2))$$

coming from

$$\begin{aligned}\frac{\partial}{\partial x}\zeta_2(s,x,(\omega_1,\omega_2)) &= \frac{\partial}{\partial x}\left(\sum_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)^{-s}\right)\\ &= -s\sum_{n_1,n_2\geq 0}(n_1\omega_1+n_2\omega_2+x)^{-s-1}\\ &= -s\zeta_2(s+1,x,(\omega_1,\omega_2)).\end{aligned}$$

Thus

$$\left(\frac{S'}{S}\right)''(x) > 0$$

in $0 < x < \omega_1 + \omega_2$ and

$$\left(\frac{S'}{S}\right)'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0.$$

This shows Figure 1. We show that $S'\left(\frac{\omega_1+\omega_2}{2}\right) < 0$. Suppose that

$$S'\left(\frac{\omega_1 + \omega_2}{2}\right) \geq 0.$$

Then

$$\frac{S'}{S}\left(\frac{\omega_1 + \omega_2}{2}\right) \geq 0$$

and

$$\frac{S'}{S}(x) \geq 0 \quad \text{in} \quad 0 < x < \omega_1 + \omega_2$$

since $\left(\frac{S'}{S}\right)'(x) > 0$ in $\frac{\omega_1+\omega_2}{2} < x < \omega_1 + \omega_2$ and $\left(\frac{S'}{S}\right)'(x) < 0$ in $0 < x < \frac{\omega_1+\omega_2}{2}$. Hence

$$S'(x) \geq 0 \quad \text{in} \quad 0 < x < \omega_1 + \omega_2.$$

Thus, especially

$$S\left(\frac{\omega_1}{2}\right) \leq S\left(\frac{\omega_1 + \omega_2}{2}\right).$$

But, this contradicts Theorem 1(1)(2): $\sqrt{2} > 1$. So we conclude that

$$S'\left(\frac{\omega_1 + \omega_2}{2}\right) < 0.$$

Thus we get Figure 2. Hence there exist uniquely α and β in $0 < \alpha < \beta < \omega_1 + \omega_2$ such that $S'(\alpha) = S'(\beta) = 0$; notice that $S(0) = 0$ and $\lim_{x \uparrow \omega_1 + \omega_2} S(x) = +\infty$. Now, inequalities

$$\frac{\omega_1}{2} \leq \alpha \leq \frac{\omega_2}{2} \quad \text{and} \quad \omega_1 + \frac{\omega_2}{2} \leq \beta \leq \omega_2 + \frac{\omega_1}{2}$$

are seen with Theorem 1(1)(3) considered. Moreover

$$S'(\omega_1 + \omega_2 - x) = -\frac{S'(x)}{S(x)^2}$$

shows that $\beta = \omega_1 + \omega_2 - \alpha$. Thus we get Theorem 2. Consequently we have Figure 3.

Proof of Theorem 3

Properties (5) and (6) follow from (1) (2) and (3) (4), respectively. From the symmetry

$$S_2(x, (\omega_1,\ \omega_2)) = S_2(x, (\omega_2,\ \omega_1))$$

and the homogeneity

$$S_2(x, (\omega_1,\ \omega_2)) = S_2(cx, (c\omega_1,\ c\omega_2))$$

for $c > 0$ (see [KK1]) we see that

$$\alpha_{\max}(\omega_1,\ \omega_2) = \alpha_{\max}(\omega_2,\ \omega_1)$$

and

$$\alpha_{\max}(\omega_1,\ \omega_2) = \frac{\alpha_{\max}(c\omega_1,\ c\omega_2)}{c}.$$

In particular

$$\alpha_{\max}\left(1, \frac{1}{t}\right) = \alpha_{\max}\left(\frac{1}{t}, 1\right) = \frac{\alpha_{\max}(1,t)}{t}.$$

Hence we find that (1) implies (2). As we also have

$$\alpha_{\min}(1,t) = 1 + t - \alpha_{\max}(1,t)$$

we see (1) implies (3) and that (2) implies (4). Therefore it suffices to show (1).

Proof of (1)

Let $0 < t < \frac{1}{6}$ and put $N(t) = [\frac{1}{6t}] + 1$. We show that

$$(N(t) - 1)t \le \alpha_{\max}(1,t) \le (N(t) + 1)t.$$

If this is proved, we have

$$\frac{1}{6} - t < \alpha_{\max}(1,t) \le \frac{1}{6} + 2t$$

from

$$\frac{1}{6t} < N(t) \le \frac{1}{6t} + 1.$$

Hence we get

$$\lim_{t \to 0} \alpha_{\max}(1,t) = \frac{1}{6}.$$

<u>*Proof of* $\alpha_{\max}(1,t) \ge (N(t)-1)t$</u>

From $0 < (N(t)-1)t \le \frac{1}{6}$, we see that

$$\frac{S_2(N(t)t,(1,t))}{S_2((N(t)-1)t,(1,t))} = \frac{1}{2\sin(\pi(N(t)-1)t)} \ge 1.$$

Hence

$$S_2(N(t)t,(1,t)) \ge S_2((N(t)-1)t,(1,t)).$$

Suppose that $(N(t)-1)t > \alpha_{\max}(1,t)$. Then

$$\begin{aligned} \alpha_{\max}(1,t) &< (N(t)-1)t \\ &< N(t)t \\ &\le \frac{1}{6} + t \\ &< \frac{1}{2}. \end{aligned}$$

So the decreasing nature of $S_2(x,(1,t))$ on $(\alpha_{\max}(1,t), \frac{1}{2})$ implies

$$S_2(N(t)t,(1,t)) < S_2((N(t)-1)t,(1,t)).$$

This gives a contradiction. Hence

$$(N(t)-1)t \le \alpha_{\max}(1,t).$$

∎

<u>*Proof of* $\alpha_{\max}(1,t) \le (N(t)+1)t$</u>

From $\frac{1}{2} > N(t)t > \frac{1}{6}$, we get

$$\frac{S_2((N(t)+1)t,(1,t))}{S_2(N(t)t,(1,t))} = \frac{1}{2\sin(\pi N(t)t)} < 1.$$

Hence

$$S_2((N(t)+1)t,(1,t)) < S_2(N(t)t,(1,t)).$$

Suppose that $(N(t)+1)t < \alpha_{\max}(1,t)$. Then the increasing nature of $S_2(x,(1,t))$ on $(0,\alpha_{\max}(1,t))$ implies

$$S_2(N(t)t,(1,t)) < S_2((N(t)+1)t,(1,t)).$$

This gives the contradiction. Hence

$$\alpha_{\max}(1,t) \leq (N(t)+1)t. \qquad \blacksquare$$

4 Proof of Theorem 4

It suffices to show (1) and (2), since (3) and (4) follow from them by the change of variable t to $\frac{1}{t}$; (5)-(8) come from (1)-(4) by

$$S_2(\alpha_{\min}(1,t),(1,t)) = S_2(\alpha_{\max}(1,t),(1,t))^{-1}.$$

To prove (1) and (2) we show the following facts: Let $0 < t < \frac{1}{6}$ and put $N(t) = [\frac{1}{6t}]+1$ as in the proof of Theorem 3. Then

(a)

$$S_2(\alpha_{\max}(1,t),(1,t)) \leq \frac{1}{\sqrt{t}4\sin(\pi t)\sin(2\pi t)}\left(6^{N(t)-2}t^{N(t)-2}(N(t)-2)!\right)^{-1},$$

(b)

$$S_2(\alpha_{\max}(1,t),(1,t)) \geq \frac{1}{\sqrt{t}}\left(\frac{1}{2}(2\pi)^{N(t)-2}t^{N(t)-2}N(t)!\right)^{-1}.$$

Moreover

(c)

$$\lim_{t\downarrow 0}\frac{(N(t)-2)\log\frac{1}{t}-(N(t)-2)\log 6-\log((N(t)-2)!)}{1/t} = \frac{1}{6},$$

(d)

$$\lim_{t\downarrow 0}\frac{(N(t)-2)\log\frac{1}{t}-(N(t)-2)\log(2\pi)-\log(N(t)!)}{1/t} = \frac{1}{6}\log\left(\frac{3e}{\pi}\right).$$

First we check that (1) and (2) follow from (a)-(d). This is easy since

$$\begin{aligned}
&\limsup_{t\downarrow 0} \frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{1/t} \\
&\underset{\text{(a)}}{\leq} \limsup_{t\downarrow 0} \frac{(N(t)-2)\log\frac{1}{t} - (N(t)-2)\log 6 - \log((N(t)-2)!) - \log(\sqrt{t}4\sin(\pi t)\sin(2\pi t))}{1/t} \\
&\underset{\text{(c)}}{=} \frac{1}{6}
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\downarrow 0} \frac{\log S_2(\alpha_{\max}(1,t),(1,t))}{1/t} \\
&\underset{\text{(b)}}{\geq} \liminf_{t\downarrow 0} \frac{(N(t)-2)\log\frac{1}{t} - (N(t)-2)\log(2\pi) - \log(N(t)!) - \log(\sqrt{t}\frac{1}{2})}{1/t} \\
&\underset{\text{(d)}}{=} \frac{1}{6}\log\left(\frac{3e}{\pi}\right).
\end{aligned}$$

Now we show (a)-(d).

Proof of (a)(b)

Using

$$S_2(\alpha_{\max}(1,t)-(l-1)t,(1,t)) = S_2(\alpha_{\max}(1,t)-lt,(1,t))S_1(\alpha_{\max}(1,t)-lt)^{-1}$$

for $l = 1, ..., N(t)-2$, we have

$$S_2(\alpha_{\max}(1,t),(1,t)) = S_2(\alpha_{\max}(1,t)-(N(t)-2)t,(1,t)) \prod_{l=1}^{N(t)-2} S_1(\alpha_{\max}(1,t)-lt)^{-1}.$$

From the inequalites

$$(N(t)-1)t \leq \alpha_{\max}(1,t) \leq (N(t)+1)t$$

shown in the proof of Theorem 3, we see that

$$t \leq \alpha_{\max}(1,t)-(N(t)-2)t \leq 3t.$$

Hence

$$S_2(\alpha_{\max}(1,t),(1,t)) \leq S_2(3t,(1,t)) \prod_{l=1}^{N(t)-2} S_1((N(t)-1-l)t)^{-1}$$

and

$$S_2(\alpha_{\max}(1,t),(1,t)) \geq S_2(t,(1,t)) \prod_{l=1}^{N(t)-2} S_1((N(t)+1-l)t)^{-1}.$$

Here we notice that

$$S_2(t,(1,t)) = \frac{1}{\sqrt{t}}$$

and

$$S_2(3t,(1,t)) = \frac{1}{\sqrt{t}4\sin(\pi t)\sin(2\pi t)}.$$

The former identity is Theorem 1(5) and the latter is obtained from

$$\begin{aligned} S_2(3t,(1,t)) &= S_2(2t,(1,t))S_1(2t)^{-1} \\ &= S_2(t,(1,t))S_1(t)^{-1}S_1(2t)^{-1}. \end{aligned}$$

Remarking

$$S_1(x) = 2\sin(\pi x) \leq 2\pi x \text{ for } 0 < x < \frac{1}{2}$$

and

$$S_1(x) = 2\sin(\pi x) \geq 6x \text{ for } 0 < x < \frac{1}{6}$$

we see that

$$\begin{aligned} \prod_{l=1}^{N(t)-2} S_1((N(t)-1-l)t) &\geq \prod_{l=1}^{N(t)-2} (6(N(t)-1-l)t) \\ &= 6^{N(t)-2}t^{N(t)-2}(N(t)-2)!, \end{aligned}$$

where we used

$$0 < (N(t)-2)t < \frac{1}{6},$$

and that

$$\begin{aligned} \prod_{l=1}^{N(t)-2} S_1((N(t)+1-l)t) &\leq \prod_{l=1}^{N(t)-2} (2\pi(N(t)+1-l)t) \\ &= \frac{1}{2}(2\pi)^{N(t)-2}t^{N(t)-2}N(t)!. \end{aligned}$$

Combining these inequalities we get (a) and (b).

Proof of (c)(d)

First, reformulate numerators of (c) and (d) as

$$
\begin{aligned}
&(N(t)-2)\log\frac{1}{t}-(N(t)-2)\log 6-\log((N(t)-2)!)\\
&\quad= (N(t)-2)\log\frac{1}{6tN(t)}+(N(t)-2)\log N(t)-\log((N(t))!)+\log(N(t)(N(t)-1))\\
&\quad= N(t)+(N(t)-2)\log\frac{1}{6tN(t)}-\log\frac{N(t)!}{N(t)^{N(t)+\frac{1}{2}}e^{-N(t)}}-\frac{5}{2}\log N(t)+\log(N(t)(N(t)-1))
\end{aligned}
$$

and

$$
\begin{aligned}
&(N(t)-2)\log\frac{1}{t}-(N(t)-2)\log(2\pi)-\log(N(t)!)\\
&\quad= (N(t)-2)\log\frac{6}{2\pi}+(N(t)-2)\log\frac{1}{t}-(N(t)-2)\log 6-\log((N(t))!)\\
&\quad= (N(t)-2)\log\frac{3}{\pi}+N(t)+(N(t)-2)\log\frac{1}{6tN(t)}-\log\frac{N(t)!}{N(t)^{N(t)+\frac{1}{2}}e^{-N(t)}}-\frac{5}{2}\log N(t).
\end{aligned}
$$

Now the estimation

$$1<6tN(t)<1+6t$$

shows that

$$
\begin{aligned}
&\lim_{t\downarrow 0}\frac{N(t)}{1/t}=\frac{1}{6},\\
&\lim_{t\downarrow 0}\log\frac{1}{6tN(t)}=0,
\end{aligned}
$$

and

$$\lim_{t\downarrow 0}\frac{\log N(t)}{1/t}=0.$$

Moreover, Stirling's formula

$$\lim_{n\to\infty}\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}=\sqrt{2\pi}$$

implies that

$$\lim_{t\downarrow 0}\frac{\log\frac{N(t)!}{N(t)^{N(t)+\frac{1}{2}}e^{-N(t)}}}{1/t}=0.$$

Hence we get (c) and (d). ∎

5 Proof of Theorem 5

(1) $(N_1, N_2) = 1$ case:

Take integers $M_1,\ M_2 \geq 1$ such that $M_1N_1 - M_2N_2 = 1$. Then

$$mM_1N_1 = m + mM_2N_2$$

and

$$\begin{aligned} S_2(mM_1N_1, (N_1, N_2)) &= S_2(m + mM_2N_2, (N_1, N_2)) \\ &= S_2(m + (mM_2 - 1)N_2, (N_1, N_2))S_1(m + (mM_2 - 1)N_2, N_1)^{-1} \\ &= S_2(m + (mM_2 - 1)N_2, (N_1, N_2))S_1\left(\frac{m + (mM_2 - 1)}{N_1}\right)^{-1} \end{aligned}$$

with

$$S_1(x) = 2\sin \pi x.$$

Hence, inductively we have

$$S_2(mM_1N_1, (N_1, N_2)) = S_2(m, (N_1, N_2)) \prod_{l=0}^{mM_2-1} S_1\left(\frac{m + lN_2}{N_1}\right)^{-1}.$$

Hence we get

$$S_2(m, (N_1, N_2)) = S_2(mM_1N_1, (N_1, N_2)) \prod_{l=0}^{mM_2-1} S_1\left(\frac{m + lN_2}{N_1}\right).$$

On the other hand

$$\begin{aligned} S_2(mM_1N_1, (N_1, N_2)) &= S_2((mM_1^{-1})N_1, (N_1, N_2))S_1((mM_1 - 1)N_1, N_2)^{-1} \\ &= S_2(N_1, (N_1, N_2)) \prod_{k=1}^{mM_1-1} S_1\left(\frac{kN_1}{N_2}\right)^{-1} \\ &= \sqrt{\frac{N_2}{N_1}} \prod_{k=1}^{mM_1-1} S_1\left(\frac{kN_1}{N_2}\right)^{-1}. \end{aligned}$$

Thus

$$S_2(m, (N_1, N_2)) = \sqrt{\frac{N_2}{N_1}} \prod_{l=0}^{mM_2-1} S_1\left(\frac{m + lN_2}{N_1}\right) \prod_{k=1}^{mM_1-1} S_1\left(\frac{kN_1}{N_2}\right)^{-1},$$

which is an algebraic number in $\mathbf{Q}\left(\sin\frac{2\pi}{N_i},\ \cos\frac{2\pi}{N_i},\ \sqrt{N_i}\ \middle|\ i=1,2\right)$.

(2) $(N_1, N_2) = 2$ case:

Since $(\frac{N_1}{2}, \frac{N_2}{2}) = 1$, the case of even m is reduced to (1) via

$$S_2(m,(N_1,N_2)) = S_2\left(\frac{m}{2},\left(\frac{N_1}{2},\frac{N_2}{2}\right)\right).$$

Now let m be an odd integer. Since either $\frac{N_1}{2}$ or $\frac{N_2}{2}$ is odd, we may assume that $\frac{N_1}{2}$ is odd without the loss of generality. Then, $m - \frac{N_1}{2}$ is even, so we may express

$$m - \frac{N_1}{2} = k_1N_1 + k_2N_2$$

for integers k_1 and k_2. Hence the algebraicity of

$$S_2(m,(N_1,N_2)) = S_2\left(\frac{N_1}{2} + k_1N_1 + k_2N_2,(N_1,N_2)\right)$$

is reduced to

$$S_2\left(\frac{N_1}{2},(N_1,N_2)\right) = \sqrt{2}$$

via the (quasi-)periodicity exactly similar to the case (1).

Proof of Examples.: Calculation of $S_2(m,(4,\ 6))$ for $m = 1, 2, ..., 9.$

From the reflection relation

$$S_2(\omega_1+\omega_2-x,\ (\omega_1,\omega_2)) = S_2(x,\ (\omega_1,\omega_2))^{-1}$$

it is sufficient to deal with the cases with $m = 1, ..., 5$. First, cases $m = 2,\ 3$ are shown from Theorem 1(2):

$$S_2\left(\frac{\omega_i}{2},\ (\omega_1,\omega_2)\right) = \sqrt{2}.$$

Secondly, cases $m = 4$ and $m = 5$ are deduced from Theorem 1(4) and (1), respectively:

$$S_2\left(\omega_1,\ (\omega_1,\omega_2)\right) = \sqrt{\frac{\omega_2}{\omega_1}}.$$

and

$$S_2\left(\frac{\omega_1+\omega_2}{2},\ (\omega_1,\omega_2)\right)=1.$$

Lastly, it remains to show the case $m=1$. Let $x=1$, $\omega_1=4$, $\omega_2=6$ in the periodicity

$$\begin{aligned} S_2(x+\omega_1,\ (\omega_1,\omega_2)) &= S_2(x,(\omega_1,\omega_2))S_1(x,\omega_2)^{-1} \\ &= S_2(x,(\omega_1,\omega_2))\left(2\sin\frac{\pi x}{\omega_2}\right)^{-1}. \end{aligned}$$

Then we have

$$\begin{aligned} S_2(5,\ (4,\ 6)) &= S_2(1,\ (4,\ 6))\left(2\sin\frac{\pi}{6}\right)^{-1} \\ &= S_2(1,\ (4,\ 6)). \end{aligned}$$

Hence

$$S_2(1,\ (4,\ 6))=S_2(5,\ (4,\ 6))=1.$$ ∎

6 Proof of Theorem 6

Notice that (1) is seen from

$$\begin{aligned} S_2(n,(1,6n)) &= S_2(n+1,(1,6n)) \\ &= \sqrt{6n}\left(\prod_{k=1}^{n}2\sin\frac{k\pi}{6n}\right)^{-1}. \end{aligned}$$

Hence, using

$$\frac{k}{n}\le 2\sin\frac{k\pi}{6n}\le\frac{k\pi}{3n}$$

for $k=1,...,n$, we have

$$\frac{n!}{n^n}\le\prod_{k=1}^{n}2\sin\frac{k\pi}{6n}\le\left(\frac{\pi}{3}\right)^n\frac{n!}{n^n}.$$

Thus

$$\sqrt{6}\frac{n^{n+\frac{1}{2}}}{n!}\left(\frac{3}{\pi}\right)^n\le S_2(n,(1,6n))\le\sqrt{6}\frac{n^{n+\frac{1}{2}}}{n!}.$$

So, the asymptotic formula of Stirling

$$\sqrt{6}\frac{n^{n+\frac{1}{2}}}{n!} \sim \sqrt{\frac{3}{\pi}}e^n$$

and

$$\sqrt{6}\frac{n^{n+\frac{1}{2}}}{n!}\left(\frac{3}{\pi}\right)^n \sim \sqrt{\frac{3}{\pi}}\left(\frac{3e}{\pi}\right)^n$$

give (2) and (3). ∎

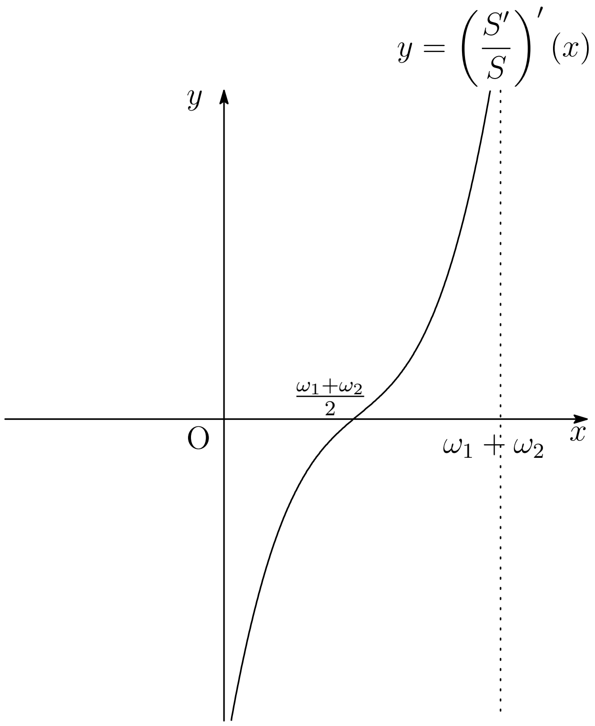

Figure 1.

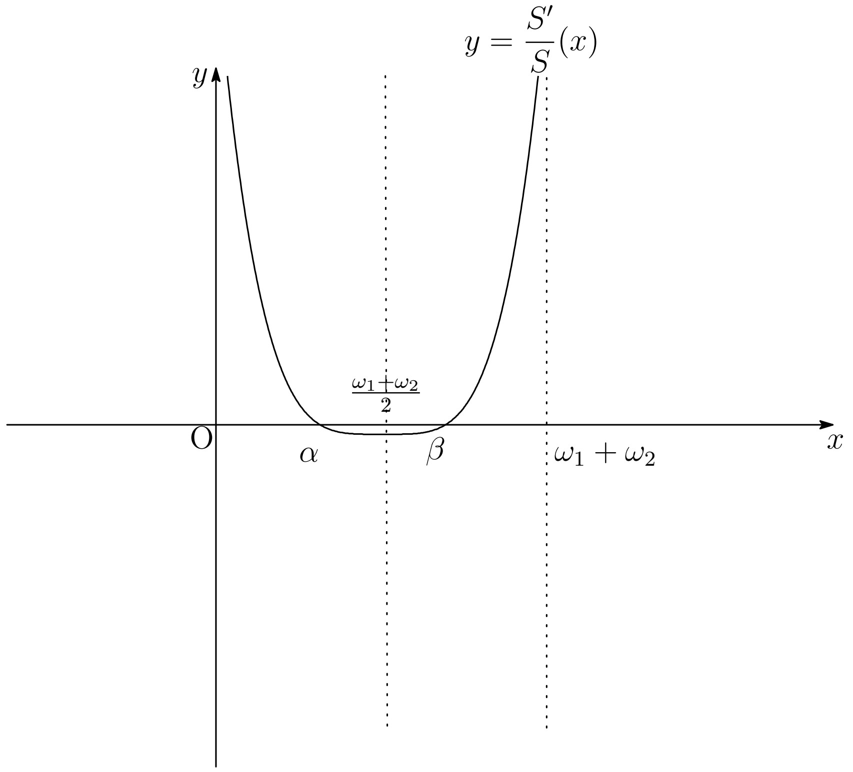

Figure 2.

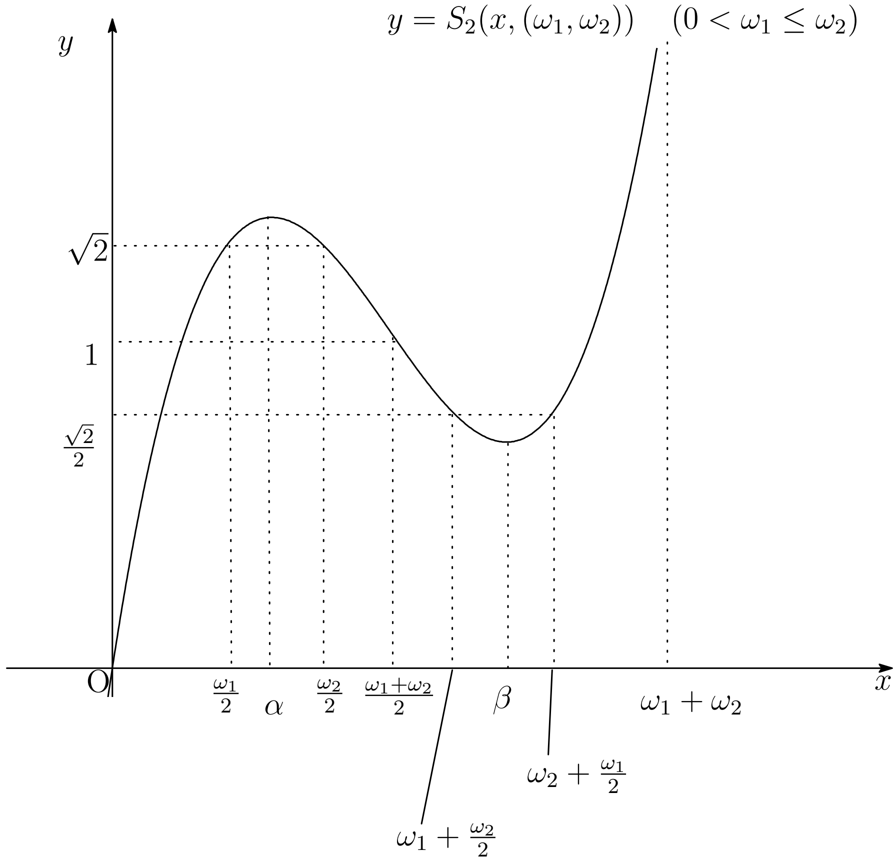

Figure 3.

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