Triple mean values of Witten L-functions

Shin-ya Koyama* & Nobushige Kurokawa†

October 14, 2015

Abstract

Mean values of Witten L-functions in the "character" aspect are investigated. After giving a general formula for mean values with the first and the second power, we explicitly calculate the cubic moment for $SU(2)$.

Key Words: mean values; Witten L-functions; Witten zeta functions

AMS Subject Classifications: 11M06, 11M41

1 Introduction

Study of mean values of zeta functions is one of central subjects in number theory. The most frequently investigated problem would be the case for the mean value of the $2k$-th power of zeta functions in the t-aspect along the critical line (the $2k$-th moment). For instance, much work has been done towards the conjectural asymptotic for the Riemann zeta function $\zeta(s)$:

$$\frac{1}{2T}\int_{-T}^{T}\zeta\left(\frac{1}{2}+it\right)^{2k}dt \sim c_k(\log T)^{k^2} \qquad (T\to\infty). \tag{1.1}$$

Analogous problems exist for various zeta functions for more general $s \in \mathbb{C}$ in more general aspects.

We find a tendency that the higher k is, the more difficult the problem is. For example, the value c_k in (1.1) as well as its proof is known only for $k = 1, 2$, which are classical results by Hardy-Littlewood [8] and Ingham [9]. The

*Department of Biomedical Engineering, Toyo University, 2100 Kujirai, Kawagoe, Saitama, 350-8585, Japan.

†Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-8551, Japan.

conjectural values of c_k are known only for $k = 3, 4$ ([3] [4]). A general form of c_k is proposed by Keating-Snaith [10] and Brezin-Hikami [2] under assuming the analogy between $\zeta(s)$ and the characteristic polynomial of random matrices.

We also observe that any odd power moment is very hard to treat. As far as the authors know, the only successful case where the cubic moment was studied is the work by Conrey and Iwaniec [5].

The goal of this paper is to deal with the mean values of the third power of Witten L-functions in the "character" aspect in $\mathrm{Re}(s) > 0$. For a compact semisimple Lie group G, Witten ([17]) defined a zeta function from the partition function of a quantum system as follows:

$$\zeta_G(s) = \sum_{\rho \in \widehat{G}} (\dim \rho)^{-s}, \tag{1.2}$$

where $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G. It is known that (1.2) is absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large ([1],[12]), and that $\zeta_G(s)$ is meromorphic in $s \in \mathbb{C}$.

In case of $G = SU(2)$, it holds that

$$\widehat{G} = \{\mathrm{Sym}^{n-1} \mid n = 1, 2, 3, ...\},$$

where Sym^{n-1} is the n-dimensional symmetric power representation defined by

$$\mathrm{Sym}^{n-1}: \ SU(2) \ni g \longmapsto \begin{pmatrix} \alpha^{n-1} & & & \\ & \alpha^{n-2}\beta & & \\ & & \ddots & \\ & & & \beta^{n-1} \end{pmatrix} \in GL(n, \mathbb{C})$$

with α and β being the eigenvalues of g. Hence

$$\zeta_{SU(2)}(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

In this sense the Witten zeta function is a generalization (a deformation) of the Riemann zeta function.

We also define the Witten L-functions after Kurokawa-Ochiai [11] by attaching "characters" which suitably twist $\rho \in \widehat{G}$. Any fixed element $g \in G$ plays the role. Namely, we regard the following map as a "character" of $\widehat{G}$:

$$g: \ \widehat{G} \ni \rho \longmapsto \frac{\mathrm{tr}(\rho(g))}{\deg \rho} \in \mathbb{C}.$$

Here $\frac{1}{\deg \rho}$ is a normalization factor so that $g(\rho) = 1$ for the unit element $g \in G$. In this manner the Witten L-function of G with a twist given by $g \in G$ is defined by

$$\zeta_G(s,g) = \sum_{\rho \in \widehat{G}} \frac{\mathrm{tr}(\rho(g))}{\deg \rho} (\deg \rho)^{-s}. \tag{1.3}$$

This is also absolutely convergent for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. It is easy to see that $\zeta_G(s,g)$ depends only on the conjugacy class of g in G.

The chief concern in this paper is to study the mean values

$$Z_G^m(s) := \int_G \zeta_G(s,g)^m dg \tag{1.4}$$

with dg the normalized Haar measure of G. Such a problem was first studied in [11] §2.8. Here $Z_G^m(s)$ means the meromorphic continuation of the defining integral (1.4). We will note in Remark (2) in §2 that it generally differs from the integral of the meromorphic continuation of $\zeta_G(s,g)^m$.

In the next section we study (1.4) for $m = 1, 2$ and $\mathrm{Re}(s) > 1$. In the final section we specialize the group as $G = SU(2)$, and calculate (1.4) for $m = 3$. We also show that the function $Z^3_{SU(2)}(s)$ is meromorphic in $s \in \mathbb{C}$.

2 Preliminary results

Proposition 2.1 (The mean value: the first power moment)**.** *For any compact semisimple Lie group G, it holds that $Z_G^1(s) = 1$.*

Proof. Let $\mathrm{Re}(s)$ be large enough so that the series (1.3) is absolutely convergent. Then

$$\begin{aligned}\int_G \zeta_G(s,g)dg &= \int_G \sum_{\rho \in \widehat{G}} \frac{\mathrm{tr}(\rho(g))}{\deg \rho} (\deg \rho)^{-s} dg \\ &= \sum_{\rho \in \widehat{G}} \frac{1}{(\deg \rho)^{s+1}} \int_G \mathrm{tr}(\rho(g))dg. \end{aligned} \tag{2.1}$$

Here we appeal to the orthogonal relation of characters. For any ρ, $\rho' \in \widehat{G}$, it holds that

$$\int_G \mathrm{tr}(\rho(g))\overline{\mathrm{tr}(\rho'(g))}dg = \begin{cases} 1 & (\rho = \rho') \\ 0 & (\text{otherwise}). \end{cases} \tag{2.2}$$

Putting ρ' to be the identity representation, we find that

$$\int_G \mathrm{tr}(\rho(g))dg = \begin{cases} 1 & (\rho = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence all terms in the sum in (2.1) are zero except for $\rho = 1$. The proposition follows from the uniqueness of the analytic continuation. $\square$

Remark. (1) The assumption on $\mathrm{Re}(s)$ is necessary. Indeed from the known fact that $\zeta_{SU(2)}(-2, g) = 0$ for all $g \in SU(2)$ ([11] Theorem1), we obviously see that $\int_{SU(2)} \zeta(-2, g)dg = 0$. We also refer to [6] on vanishing of Witten zeta functions at $s = -2$ for the compact p-adic Lie groups. (See also [14], [16] for some generalizations.)

(2) The preceding remark shows that a meromorphic continuation of (1.4) is different from the integral of the meromorphically continued $\zeta_G(s, g)^m$. Thus analyticity of $Z_G^m(s)$ is nontrivial, in the sense that it is not an immediate consequence from that of $\zeta_G(s, g)$.

In the next proposition we assume that the group G satisfies the following condition:

$$\mathrm{tr}(\rho(g)) \in \mathbb{R} \qquad (\forall \rho \in \widehat{G},\ \forall g \in G). \tag{$*$}$$

The special unitary group $G = SU(2)$ is an example of such G, as shown in Proposition 3.1 below.

Proposition 2.2 (The double mean value: the square moment)**.** *For any compact semisimple Lie group G satisfying the condition $(*)$, it holds that*

$$Z_G^2(s) := \int_G \zeta_G(s, g)^2 dg = \zeta_G(2s + 2),$$

where $\mathrm{Re}(s)$ *is sufficiently large.*

Proof. Let $\mathrm{Re}(s)$ be large enough so that the series (1.3) is absolutely convergent. Then

$$\begin{aligned}\int_G \zeta_G(s, g)^2 dg &= \int_G \sum_{\rho_1, \rho_2 \in \widehat{G}} \frac{\mathrm{tr}(\rho_1(g))\,\mathrm{tr}(\rho_2(g))}{(\deg \rho_1)^{s+1}(\deg \rho_2)^{s+1}} dg \\ &= \sum_{\rho_1, \rho_2 \in \widehat{G}} \frac{1}{(\deg \rho_1)^{s+1}(\deg \rho_2)^{s+1}} \int_G \mathrm{tr}(\rho_1(g))\,\mathrm{tr}(\rho_2(g))dg.\end{aligned}$$

By the orthogonal relation (2.2), all terms in the sum are zero except for $\rho_1 = \rho_2$. Therefore if we put $\rho_1 = \rho_2 = \rho$, it holds that

$$\int_G \zeta_G(s, g)^2 dg = \sum_{\rho \in \widehat{G}} \frac{1}{(\deg \rho)^{2s+2}} = \zeta_G(2s + 2).$$

□

When G is a finite group, we have

$$Z_G^m(s) = \frac{1}{|G|} \sum_{g \in G} \zeta_G(s, g)^m.$$

The following proposition is an example of calculation of mean values for general $m \geq 1$.

Proposition 2.3 (Mean values $Z^m_{S_3}(s)$)**.** *Let $G = S_3$ be the symmetric group of degree 3. Then it holds that*

$$Z^m_{S_3}(s) = \frac{(2+2^{-s})^m + 2(2-2^{-s-1})^m}{6}.$$

Especially,

$$\begin{aligned} Z^1_{S_3}(s) &= 1, \\ Z^2_{S_3}(s) &= 2 + 2^{-2s-2} = \zeta_{S_3}(2s+2). \end{aligned}$$

Proof. The Witten L-function $\zeta_G(s, g)$ depends only on the conjugacy class of g. Conjugacy classes of symmetric groups are classified by the cycle type. The elements in S_3 consist of two cyclic permutations of order three, three transpositions of order two, and the identity. Hence

$$Z^m_{S_3}(s) = \frac{1}{6}\left(2\zeta_{S_3}(s, (1\,2\,3))^m + 3\zeta_{S_3}(s, (1\,2))^m + \zeta_{S_3}(s, (1))^m\right).$$

Now we will calculate $\zeta_{S_3}(s, g)$ for each $g \in S_3$. We have $\widehat{S_3} = \{\rho_1, \rho_2, \rho_3\}$ with ρ_1 the trivial representation, ρ_2 the signature, and ρ_3 the unique two dimensional irreducible representation defined by

$$\rho_3((1\,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_3((1\,2\,3)) = \begin{pmatrix} e^{\frac{2}{3}\pi i} & 0 \\ 0 & e^{-\frac{2}{3}\pi i} \end{pmatrix},$$

whose traces are 0 and -1, respectively. Then each L-function is given by

$$\begin{aligned} \zeta_{S_3}(s, (1)) &= 1 + 1 + \frac{2}{2^{1+s}} = 2 + 2^{-s}, \\ \zeta_{S_3}(s, (1\,2)) &= 1 - 1 + 0 \cdot 2^{-s} = 0, \\ \zeta_{S_3}(s, (1\,2\,3)) &= 1 + 1 + \frac{-1}{2^{1+s}} = 2 - 2^{-s-1}. \end{aligned}$$

Therefore we get the conclusion. □

The characters attached to Witten L-functions are generalized to convolutions of m times of twists ($m = 1, 2, 3, \ldots$):

$$\zeta_G(s, (g_1, \ldots, g_m)) = \sum_{\rho \in \widehat{G}} \frac{\mathrm{tr}(\rho(g_1)) \cdots \mathrm{tr}(\rho(g_m))}{(\deg \rho)^{s+m}}.$$

We can consider another type of mean values as

$$\widetilde{Z}^m_G(s) := \int_G \zeta_G(s, (\underbrace{g, \ldots, g}_{m}))dg = \int_G \sum_{\rho \in \widehat{G}} \frac{(\mathrm{tr}(\rho(g)))^m}{(\deg \rho)^{s+m}} dg. \tag{2.3}$$

This is calculated for $G = S_3$ in the following proposition.

Proposition 2.4. *Let S_3 be the symmetric group of degree 3. Then it holds that*

$$\widetilde{Z}^m_{S_3}(s) = \int_{S_3} \zeta_{S_3}(s, (\underbrace{g, ..., g}_{m}))dg = \frac{3+(-1)^m}{2} + \frac{2^{-1}+(-1)^m}{3} \cdot 2^{-s-m}.$$

Proof. We again use the classification of the conjugacy classes of S_3 and the explicit form of elements in $\widehat{S_3}$ given in the proof of the preceding proposition. It holds that

$$\begin{aligned}
&\int_{S_3} \zeta_{S_3}(s, (\underbrace{g, ..., g}_{m}))dg \\
&= \frac{1}{6} \sum_{\rho \in \widehat{S_3}} \left(\frac{2\,\mathrm{tr}(\rho(1\,2\,3))^m}{(\deg \rho)^{s+m}} + \frac{3\,\mathrm{tr}(\rho(1\,2))^m}{(\deg \rho)^{s+m}} + \frac{1}{(\deg \rho)^{s+m}} \right) \\
&= \frac{1}{6} \left(\underbrace{2+3+1}_{\rho_1} + \underbrace{2+3(-1)^m+1}_{\rho_2} + \underbrace{\frac{2(-1)^m}{2^{s+m}} + \frac{3 \cdot 0}{2^{s+m}} + \frac{1}{2^{s+m}}}_{\rho_3} \right) \\
&= \frac{3+(-1)^m}{2} + \frac{2^{-1}+(-1)^m}{3} 2^{-s-m}.
\end{aligned}$$

□

3 Triple mean values: the cubic moment

Proposition 3.1. *The group $G = SU(2)$ satisfies the condition $(*)$.*

Proof. We put the eigenvalues of $g \in SU(2)$ as $e^{\pm i\theta}$. The conjugacy classes of $G = SU(2)$ are parametrized by $\theta \in [0, \pi]$. Since the Witten L-function depends only on the conjugacy class of $g \in G$, we denote $\zeta_G(s, g) = \zeta_G(s, [\theta])$. Then we compute

$$\begin{aligned}
\operatorname{tr}(\operatorname{Sym}^{n-1}(g)) &= \operatorname{tr}\left(\operatorname{Sym}^{n-1}\left(\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix}\right)\right) \\
&= \operatorname{tr}\begin{pmatrix} e^{i(n-1)\theta} & & & \\ & e^{i(n-3)\theta} & & \\ & & \ddots & \\ & & & e^{-i(n-1)\theta}\end{pmatrix} \\
&= \begin{cases} n & (\theta = 0) \\ \dfrac{\sin(n\theta)}{\sin\theta} & (0 < \theta < \pi) \\ (-1)^{n-1}n & (\theta = \pi). \end{cases} \qquad (3.1)
\end{aligned}$$

□

From (3.1) we obtain the following corollary immediately.

Corollary 3.2 (The explicit form of $\zeta_{SU(2)}(s, [\theta])$). *We have in* $\operatorname{Re}(s) > 1$ *that*

$$\zeta_{SU(2)}(s, [\theta]) = \begin{cases} \zeta(s) & (\theta = 0) \\ \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{\sin\theta} n^{-s-1} & (0 < \theta < \pi) \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s) & (\theta = \pi). \end{cases}$$

Here the second expression is absolutely convergent by the rough estimate $\sin(n\theta)/\sin\theta = O(n)$ as $n \to \infty$ (see, for example, [7] 1.331.1).

Before calculationg the triple mean value, we are introducing preliminary calculations which are good for general m. For $G = SU(2)$ and $\operatorname{Re}(s) > 1$, we compute by putting $\rho_j = \operatorname{Sym}^{n_j - 1}$ that

$$\begin{aligned}
&\int_G \zeta_G(s, g)^m dg \\
&= \sum_{\rho_1, \ldots, \rho_m \in \widehat{G}} \frac{1}{((\deg \rho_1) \cdots (\deg \rho_m))^{s+1}} \int_G \operatorname{tr}(\rho_1(g)) \cdots \operatorname{tr}(\rho_m(g)) dg \\
&= \sum_{n_1, \ldots, n_m \geq 1} \frac{1}{(n_1 \cdots n_m)^{s+1}} \int_G \operatorname{tr}(\operatorname{Sym}^{n_1 - 1}(g)) \cdots \operatorname{tr}(\operatorname{Sym}^{n_m - 1}(g)) dg.
\end{aligned}$$

The last integrand depends only on the conjugacy class of g. We use the notation

$[\theta]$ defined in the proof of Proposition 3.1. Then the last integral equals by (3.1)

$$\begin{aligned}
&\int_0^\pi \operatorname{tr}\left(\operatorname{Sym}^{n_1-1}([\theta])\right)\cdots\operatorname{tr}\left(\operatorname{Sym}^{n_m-1}([\theta])\right)\frac{2}{\pi}\sin^2\theta d\theta \\
&= \int_0^\pi \frac{\sin(n_1\theta)}{\sin\theta}\cdots\frac{\sin(n_m\theta)}{\sin\theta}\frac{2}{\pi}\sin^2\theta d\theta \\
&= \frac{2}{\pi}\int_0^\pi \frac{\sin(n_1\theta)\cdots\sin(n_m\theta)}{\sin^{m-2}\theta}d\theta. \qquad (3.2)
\end{aligned}$$

Putting it by $c(n_1, ..., n_m)$, we have for $\operatorname{Re}(s) > 1$ that

$$Z_G^m(s) := \int_G \zeta_G(s,g)^m dg = \sum_{n_1,...,n_m\geq 1}\frac{c(n_1,...,n_m)}{(n_1\cdots n_m)^{s+1}}. \qquad (3.3)$$

Theorem 3.3 (Triple mean values: the cubic moment)**.** *It holds for $G = SU(2)$ in $\operatorname{Re}(s) > 0$ that*

$$\begin{aligned}
Z_G^3(s) &:= \int_G \zeta_G(s,g)^3 dg \\
&= \sum_{m_1,m_2,m_3\geq 0}\left((m_1+m_2+1)(m_2+m_3+1)(m_3+m_1+1)\right)^{-s-1}.
\end{aligned} \qquad (3.4)$$

This function is meromorphically continued to all $s \in \mathbb{C}$.

Proof. We compute (3.3) for $m = 3$. We first calculate the case $\operatorname{Re}(s) > 1$, where the expression in Corollary 3.2 is valid. By transforming the product of the sine function into sums, we have

$$\begin{aligned}
&c(n_1, n_2, n_3) \\
&= \frac{2}{\pi}\int_0^\pi \frac{\sin(n_1\theta)\sin(n_2\theta)\sin(n_3\theta)}{\sin\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^\pi \left(\frac{\sin((n_1+n_2-n_3)\theta)}{\sin\theta} + \frac{\sin((n_2+n_3-n_1)\theta)}{\sin\theta}\right. \\
&\qquad\qquad \left. + \frac{\sin((n_3+n_1-n_2)\theta)}{\sin\theta} - \frac{\sin((n_1+n_2+n_3)\theta)}{\sin\theta}\right)d\theta \\
&= \frac{A(n_1+n_2-n_3)+A(n_2+n_3-n_1)+A(n_3+n_1-n_2)-A(n_1+n_2+n_3)}{2}
\end{aligned} \qquad (3.5)$$

with

$$A(n) := \frac{1}{\pi}\int_0^\pi \frac{\sin(n\theta)}{\sin\theta}d\theta.$$

We first compute $A(n)$ for $n \geq 1$. It holds that

$$\begin{aligned} A(n) &= \frac{1}{\pi}\int_0^\pi \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} d\theta \\ &= \frac{1}{\pi}\int_0^\pi \left(e^{i(n-1)\theta} + e^{i(n-3)\theta} + \cdots + e^{-i(n-1)\theta} \right) d\theta \\ &= \begin{cases} 1 & (n: \text{ odd}) \\ 0 & (n: \text{ even}), \end{cases} \end{aligned}$$

because the integrand contains the constant term "1" if and only if n is odd, which contribute 1 to $A(n)$, and the integral vanishes for all other terms. Since $A(n)$ is an odd function in n, we eventually have for $n \in \mathbb{Z}$ that

$$A(n) = \begin{cases} \operatorname{sgn}(n) & (n: \text{ odd}) \\ 0 & (n: \text{ even}). \end{cases}$$

Next we calculate (3.5). When $n_1+n_2+n_3$ is even, all of $n_1+n_2-n_3$, $n_2+n_3-n_1$ and $n_3+n_1-n_2$ are even, and thus $c(n_1, n_2, n_3) = 0$. Assume that $n_1+n_2+n_3$ is odd. Then all of $n_1+n_2-n_3$, $n_2+n_3-n_1$ and $n_3+n_1-n_2$ are odd, and

$$c(n_1, n_2, n_3) = \frac{\operatorname{sgn}(n_1+n_2-n_3) + \operatorname{sgn}(n_2+n_3-n_1) + \operatorname{sgn}(n_3+n_1-n_2) - 1}{2}.$$

Here we put the following condition as $(**)$:

All of $n_1+n_2-n_3$, $n_2+n_3-n_1$ and $n_3+n_1-n_2$ are positive,
and $n_1+n_2+n_3$ is odd. $\quad (**)$

When the triple (n_1, n_2, n_3) satisfies $(**)$, we have $c(n_1, n_2, n_3) = 1$. Assume that the triple (n_1, n_2, n_3) does not satisfy $(**)$. Then it is easy to see that only one of $n_1+n_2-n_3$, $n_2+n_3-n_1$ and $n_3+n_1-n_2$ is negative or zero. But it cannot be zero, because it is odd by assumption. So one of $n_1+n_2-n_3$, $n_2+n_3-n_1$ and $n_3+n_1-n_2$ is negative, and the other two are positive. Hence we conclude that $c(n_1, n_2, n_3) = \frac{1+1-1-1}{2} = 0$, when the triple (n_1, n_2, n_3) does not satisfy $(**)$. Therefore we successfully have the final form of the coefficients as

$$c(n_1, n_2, n_3) = \begin{cases} 1 & (**) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.6}$$

By (3.3) the triple mean value is

$$Z_G^3(s) = \int_G \zeta_G(s, g)^3 dg = \sum_{\substack{n_1, n_2, n_3 \geq 1 \\ (**)}} \frac{1}{(n_1 n_2 n_3)^{s+1}}. \tag{3.7}$$

Next we will see that the right hand side of (3.7) is absolutely convergent in $\mathrm{Re}(s) > 0$. It is easy once we notice that its absolute value is $O\left(\zeta(\mathrm{Re}(s)+1)^3\right)$. Therefore $Z_G^3(s)$ is analytically continued to $\mathrm{Re}(s) > 0$ by (3.7).

In what follows we rewrite (3.7) to a simpler form. Put

$$m_3 := \frac{n_1+n_2-n_3-1}{2}, \quad m_1 := \frac{n_2+n_3-n_1-1}{2}, \quad m_2 := \frac{n_3+n_1-n_2-1}{2}.$$

Then there is one-to-one correspondence between the set of all triples (n_1, n_2, n_3) with $(**)$ and the set of all triples $(m_1, m_2, m_3) \in (\mathbb{Z}_{\geq 0})^3$. The inverse correspondence

$$n_1 = m_2 + m_3 - 1, \quad n_2 = m_3 + m_1 - 1, \quad n_3 = m_1 + m_2 - 1$$

leads to the conclusion.

The meromorphic continuation was generally shown by Mellin [13]. $\square$

By this theorem the first few terms of $Z_G^3(s)$ turns to be as follows:

$$Z_G^3(s) = 1 + \frac{3}{4^s} + \frac{3}{9^s} + \frac{3}{12^s} + \frac{3}{16^s} + \frac{6}{24^s} + \frac{3}{25^s} + \frac{1}{27^s} + \cdots.$$

We can also compute the other type (2.3) of triple mean value for $G = SU(2)$ and $m = 3$.

Theorem 3.4. *It holds for* $\mathrm{Re}(s) > -2$ *that*

$$\widetilde{Z}^3_{SU(2)}(s) = \int_{SU(2)} \zeta_{SU(2)}(s, (g, g, g))dg = (1 - 2^{-s-3})\zeta(s+3).$$

In particular, the function $\widetilde{Z}^3_{SU(2)}(s)$ *is meromorphic on* $\mathbb{C}$.

Proof. We compute

$$\begin{aligned}\int_{SU(2)} \zeta_{SU(2)}(s, (g, g, g))dg &= \sum_{n=1}^{\infty} \frac{1}{n^{s+3}} \int_0^{\pi} \left(\frac{\sin n\theta}{\sin\theta}\right)^3 \frac{2}{\pi} \sin^2\theta d\theta \\ &= \sum_{n=1}^{\infty} \frac{c(n,n,n)}{n^{s+3}}, \end{aligned} \tag{3.8}$$

where $c(n_1, n_2, n_3)$ is defined by (3.2). By applying (3.6) with $m = 3$ and $n_1 = n_2 = n_3$, we find that

$$c(n,n,n) = \begin{cases} 1 & (n: \text{ odd}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then (3.8) is absolutely convergent in $\mathrm{Re}(s) > -2$ and it holds that

$$\int_{SU(2)} \zeta_{SU(2)}(s,(g,g,g))dg = \sum_{\substack{n\geq 1 \\ \text{odd}}} \frac{1}{n^{s+3}} = (1-2^{-s-3})\zeta(s+3).$$

$\square$

Remark (Quadruple case)**.** The case of $m = 4$ is also calculated as follows. Starting from the identity

$$\begin{aligned}
\widetilde{Z}^4_{SU(2)}(s) &= \int_{SU(2)} \zeta_{SU(2)}(s,(g,g,g,g))dg \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{s+4}} \int_0^{\pi} \left(\frac{\sin n\theta}{\sin\theta}\right)^4 \frac{2}{\pi} \sin^2\theta d\theta \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{s+4}} \cdot \frac{2}{\pi} \int_0^{\pi} \frac{\sin^4 n\theta}{\sin^2\theta} d\theta,
\end{aligned}$$

the last integral is calculated as follows

$$\begin{aligned}
&\frac{2}{\pi} \int_0^{\pi} \frac{\sin^4 n\theta}{\sin^2\theta} d\theta \\
&= \frac{-1}{2\pi} \int_0^{\pi} \left(\frac{e^{in\theta}-e^{-in\theta}}{e^{i\theta}-e^{-i\theta}}(e^{in\theta}-e^{-in\theta})\right)^2 d\theta \\
&= \frac{-1}{2\pi} \int_0^{\pi} \left((e^{i(n-1)\theta}+e^{i(n-3)\theta}+\cdots+e^{-i(n-1)\theta})(e^{in\theta}-e^{-in\theta})\right)^2 d\theta \\
&= \frac{-1}{2\pi} \int_0^{\pi} \left((e^{i(2n-1)\theta}+e^{i(2n-3)\theta}+\cdots+e^{i\theta})-(e^{-i\theta}+e^{-3i\theta}+\cdots+e^{-i(2n-1)\theta})\right)^2 d\theta \\
&= \frac{-1}{2\pi} \int_0^{\pi} (-2n+[\text{nonconstant terms}])d\theta = n.
\end{aligned}$$

For the nonconstant terms are written as a linear combination of

$$e^{iN\theta}+e^{-iN\theta} = 2\cos(N\theta) \qquad (N \in \mathbb{Z}\setminus\{0\}),$$

whose integral vanishes as

$$\int_0^{\pi} \cos(N\theta)d\theta = 0 \qquad (\forall N \in \mathbb{Z}\setminus\{0\}).$$

Therefore we conclude that

$$\widetilde{Z}^4_{SU(2)}(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+4}} \cdot n = \zeta(s+3). \tag{3.9}$$

This is valid for $\mathrm{Re}(s) > -2$, and it shows that $\widetilde{Z}^4_{SU(2)}(s)$ has an analytic continuation to the entire plane except for a simple pole at $s = -2$.

We can directly calculate that $\int_{SU(2)} \zeta_{SU(2)}(-2,(g,g,g,g))dg = \infty$ by using the result of Min [14] on the values $\zeta_{SU(2)}(-2,(g,g,g,g))$. This happens to agree to our conclusion (3.9) that $\widetilde{Z}^4_{SU(2)}(s)$ has a pole at $s = -2$. Although the problems are different as noted in Remark (2) after Proposition 2.1, reasoning of this coincidence may be an interesting problem for future study.

4 Generalizations

Let G be as above, and H be a subgroup of G. Mean values are generalized to the average over H as

$$Z^m_{G,H}(s) := \int_H \zeta_G(s,h)^m dh$$

and

$$\widetilde{Z}^m_{G,H}(s) := \int_H \zeta_G(s,(\underbrace{h,...,h}_{m}))dh.$$

It is easy to see that $Z^m_{G,\{1\}}(s) = \zeta_G(s)^m$ and $Z^m_{G,G}(s) = Z^m_G(s)$.

Theorem 4.1. *Let G and H be a pair of compact semisimple Lie groups such that $G = \underbrace{H \times \cdots \times H}_{m}$. We regard H as a subgroup of G by diagonal embedding. Then the following identities hold.*

(1) $\zeta_G(s,(\underbrace{h,...,h}_{m})) = \zeta_H(s,h)^m$.

(2) $\widetilde{Z}^m_{G,H}(s) = Z^m_H(s)$.

Proof. (1) The map

$$\underbrace{\widehat{H} \times \cdots \times \widehat{H}}_{m} \ni (\rho_1,...,\rho_m) \longmapsto \rho_1 \boxtimes \cdots \boxtimes \rho_m \in \widehat{G}$$

defined by

$$(\rho_1 \boxtimes \cdots \boxtimes \rho_m)(h_1,...,h_m) := \rho_1(h_1) \otimes \cdots \otimes \rho_m(h_m)$$

is an isomorphism. It also holds that

$$\deg(\rho_1 \boxtimes \cdots \boxtimes \rho_m) = (\deg \rho_1) \cdots (\deg \rho_m).$$

Hence

$$\zeta_G(s, (\underbrace{h, ..., h}_{m})) = \sum_{\rho \in \widehat{G}} \frac{\mathrm{tr}(\rho(h, ..., h))}{(\deg \rho)^{s+1}}$$
$$= \sum_{\rho_1, \ldots, \rho_m \in \widehat{H}} \frac{\mathrm{tr}(\rho_1(h)) \cdots \mathrm{tr}(\rho_m(h))}{((\deg \rho_1) \cdots (\deg \rho_m))^{s+1}}$$
$$= \left(\sum_{\rho \in \widehat{H}} \frac{\mathrm{tr}(\rho(h))}{(\deg \rho)^{s+1}} \right)^m = \zeta_H(s, h)^m.$$

(2) By (1), we have

$$\widetilde{Z}^m_{G,H}(s) = \int_H \zeta_G(s, (\underbrace{h, ..., h}_{m}))dh = \int_H \zeta_H(s, h)^m dh = Z^m_H(s).$$

$\square$

By Theorem 3.3, the following theorem is immediate.

Theorem 4.2. *Put $G = SU(2) \times SU(2) \times SU(2)$, and let $H = SU(2)$ be a subgroup of G embedded diagonally. Then $\widetilde{Z}^3_{G,H}(s)$ is explicitly expressed by the Dirichlet series* (3.4) *in* $\mathrm{Re}(s) > 0$*, and has a meromorphic continuation to the whole plane* $\mathbb{C}$.

References

[1] L. Bartholodi and P. de la Harpe, *Representation zeta functions of wreath products with finite groups*, Groups Geom. Dyn. **4** (2010), 209-249.

[2] E. Brézin and S. Hikami, *Characteristic polynomials of random matrices*, Comm. Math. Phys. **214** (2000), 111-135.

[3] J.B. Conrey and A. Ghosh, *Mean values of the Riemann zeta-function. III*, Proceedings of the Amalfi Conference on Analytic Number Theory, Univ. Salerno, Salerno, 1992, pp. 35–59.

[4] J.B. Conrey and S.M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107** (2001), 577-604.

[5] J.B. Conrey and H. Iwaniec, *The cubic moment of central values of automorphic L-functions*, Ann. of Math. **151** (2000), 1175-1216.

[6] J. González-Sánchez, A. Jaikin-Zapirain, and B. Klopsch, *The representation zeta function of a FAb compact p-adic Lie group vanishes at* -2., Bull. London Math. Soc. **46** (2014), 239-244.

[7] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products, sixth edition*, Academic Press, 2000.

[8] G.H. Hardy and J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41** (1918), 119-196.

[9] A.E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. (2) **27** (1926), 273-300.

[10] J.P. Keating and N.C. Snaith, *Random matrix theory and* $\zeta(1/2+it)$, Comm. Math. Phys. **214** (2000), 57-89.

[11] N. Kurokawa and H. Ochiai, *Zeros of Witten zeta functions and absolute limit*, Kodai Math. J. **36** (2013), 440-454.

[12] M. Larsen and A. Lubotzky, *Representation growth of linear groups*, J. Eur. Math. Soc. **10** (2008), 351-390.

[13] H. Mellin, *Eine Formel für den Logarithmus transcendenter Funktionen von endlichen Geschlecht*, Acta Soc. Scient. Fennicae **29** (1900), 3-49.

[14] J. Min, *Zeros and special values of Witten zeta functions and Witten L-functions*, J. Number Theory **134** (2014), 240-257.

[15] ———, *Vanishing of Witten L-functions for* S_n (2015).

[16] ———, *Mean values of the Witten L-function for* $SU(2)$ (2015).

[17] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), 153-209.