# Absolute Modular Forms

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**Abstract.** We describe examples of absolute modular forms coming from differentiations of multiple sine functions. We give an identity in weight 3 indicating the graded structure.

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## 1 Introduction

The absolute modular form is a new kind of modular forms. We say that a (holomorphic) function $f$ on

$$\mathcal{D}_r = \left\{ (u_1, \cdots, u_r) \in \mathbf{C}^r \;\middle|\; \begin{array}{c} u_1, \cdots, u_r \text{ and 1 belong to one side} \\ \text{with respect to a line crossing 0} \end{array} \right\}$$

is an absolute modular form of weight $k$ if it satisfies the following two conditions:

(1) $f(u_1, \cdots, u_r)$ is symmetric,

(2) $f(\frac{1}{u_1}, \frac{u_2}{u_1}, \cdots, \frac{u_r}{u_1}) = u_1^k f(u_1, u_1, \cdots, u_r)$.

The Eisenstein series

$$e_k(u_1, \cdots, u_r) = \sum_{n_1, \cdots, n_{r+1} \geq 0}{}' (n_1 u_1 + \cdots + n_r u_r + n_{r+1})^{-k} \qquad (k > r+1) \tag{1.1}$$

is a typical example, which is analogous to the classical modular forms. A strict difference is that the sum is taken over a semi-lattice here, while the classical sum was taken over a whole lattice.

We notice the shapes of $\mathcal{D}_r$ for $r = 1, 2$ as

$$\mathcal{D}_1 = \mathbf{C} - \mathbf{R}_{\leq 0} = \{u \in \mathbf{C} \mid -\pi < \arg(u) < \pi\}$$

and

$$\mathcal{D}_2 = \{(u, v) \in \mathcal{D}_1^2 \mid -\pi < \arg(u) - \arg(v) < \pi\}.$$

The absolute modular form is also called the Stirling modular form, since it was originated in an old paper of Barnes [B] (p.397), where a function $\rho_r(\omega_1, \cdots, \omega_r)$ was called an "absolute modular form" associated to the multiple gamma function (the name "absolute" is coming from the Stirling (asymptotic) formula for $n!$ as $n \to \infty$) (cf. [K3, K4, K5, KK3, Ko]): we refer to [KOW] and [CCM] for mathematics over $\mathbf{F}_1$. The absolute modula form is a function of the semi-lattice

$$\mathbf{Z}_{\geq 0}\omega_1 + \cdots + \mathbf{Z}_{\geq 0}\omega_r.$$

The absolute modular group is identified as

$$GL_r(\mathbf{F}_1) = S_r = \mathrm{Aut}(\mathbf{Z}_{\geq 0}\omega_1 + \cdots + \mathbf{Z}_{\geq 0}\omega_r).$$

From this original viewpoint, we may call a function $F$ on

$$\mathbf{D}_r = \left\{ (\omega_1, \cdots, \omega_r) \in \mathbf{C}^r \;\middle|\; \begin{array}{l} \omega_1, \cdots, \omega_r \text{ belong to one side} \\ \text{with respect to a line crossing } 0 \end{array} \right\}$$

as an absolute modular form if it satisfies the following two conditions

(1) $F(\omega_1, \cdots, \omega_r)$ is symmetric,

(2) $F(\omega_1, \cdots, \omega_r)$ is homogeneous of degree $-k$: $F(c\omega_1, \cdots, c\omega_r) = c^{-k}F(\omega_1, \cdots, \omega_r)$ for $c \in \mathbf{C} \setminus \{0\}$.

We remark that $\rho_r(\omega_1, \cdots \omega_r)$ is a complicated function and it is rather difficult to make a general theory containing it. For example, $\rho_r(\omega_1, \cdots, \omega_r)$ satisfies (1) above, but unfortunately we need a homogeneous function $k = k(\omega_1, \cdots, \omega_r)$ for (2). So we do not treat $\rho_r(\omega_1, \cdots, \omega_r)$ directly in this paper. (But, see Theorem 6 below.)

It is useful to compare the situation with the case of ordinary modular forms which are considered to be functions of the lattice $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. In that case we usually regard such functions as functions in $\tau = \omega_2/\omega_1$ on the upper (or lower) half plane to simplify the treatment. Here the modular group is obtained as

$$GL_2(\mathbf{Z}) = \mathrm{Aut}(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2).$$

Similarly we associate a function $f: \mathcal{D}_r \to \mathbf{C}$ to $F: \mathbf{D}_{r+1} \to \mathbf{C}$ under

$$f(u_1, \cdots, u_r) = F(u_1, \cdots, u_r, 1)$$

and

$$F(\omega_1, \cdots, \omega_{r+1}) = \omega_{r+1}^{-k} f\left(\frac{\omega_1}{\omega_{r+1}}, \cdots, \frac{\omega_r}{\omega_{r+1}}\right).$$

It is easy to see that the conditions (1)(2) on $f$ and $F$ are equivalent under this correspondence.

Thus, hereafter we deal with functions $f$ on $\mathcal{D}_r$ of weight $k \geq 0$. (Of course it is advisable to keep in mind the functions $F$ on $\mathbf{D}_{r+1}$ as well.) In this paper we pick up two constructions:

(a) $\mathscr{S}_k(u_1, \cdots, u_r) = S_{r+1}^{(k)}(0, (u_1, \cdots, u_r, 1))$ for $k \geq 0$.

(b) $e_k(u_1, \cdots, u_r) = \zeta_{r+1}(k, (u_1, \cdots, u_r, 1))$ for $k \geq 0$.

Here, $S_r(x, (\omega_1, \cdots, \omega_r))$ is the multiple sine function defined as

$$S_r(x, (\omega_1, \cdots, \omega_r)) = \prod_{n_1, \cdots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x) \left( \prod_{m_1, \cdots, m_r \geq 1} (m_1\omega_1 + \cdots + m_r\omega_r - x) \right)^{(-1)^{r-1}}$$

with the regularized product notation $\prod$ due to Deninger [D]:

$$\prod_{\lambda} \lambda = \exp\left( - \frac{d}{ds} \sum_{\lambda} \lambda^{-s} \Big|_{s=0} \right).$$

Using the multiple gamma function

$$\Gamma_r(x, (\omega_1, \cdots, \omega_r)) = \left( \prod_{n_1, \cdots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x) \right)^{-1}$$

we can express the multiple sine function as

$$S_r(x, (\omega_1, \cdots, \omega_r)) = \Gamma_r(x, (\omega_1, \cdots, \omega_r))^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x, (\omega_1, \cdots, \omega_r))^{(-1)^r}.$$

We refer to [K1] [K2] [KK1] for a theory of multiple sine functions; see Manin [M] for an excellent survey.

The other notation $\zeta_r(s, (\omega_1, \cdots, \omega_r))$ is a kind of multiple version of the Riemann (or Hurwitz) zeta function

$$\zeta_r(s, (\omega_1, \cdots, \omega_r)) = \sideset{}{'}\sum_{n_1, \cdots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r)^{-s}.$$

We remark that $e_k(u_1, \cdots, u_r)$, which is given by (1.1) for $k > r+1$, is defined by the analytically continued $\zeta_r(k, (u_1, \cdots, u_r, 1))$ for $k \leq r+1$. Hence, we can consider $e_0(u_1, \cdots, u_r)$ for example.

We find a neat expression (not mentioned in [B]) for $\rho_r(\omega_1, \cdots, \omega_r)$:

$$\begin{aligned} \rho_r(\omega_1, \cdots, \omega_r) &= \sideset{}{'}\prod_{n_1, \cdots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r) \\ &= \exp\left(-\zeta_r'(0, (\omega_1, \cdots, \omega_r))\right). \end{aligned}$$

Hereafter, it would be suggestive to regard $e_k$ and $\mathscr{S}_k$ as "(generalized) Eisenstein series" and "cusp forms," respectively. The present state of our experience on absolute modular forms is primitive, so we must postpone developing the general theory to future papers.

We also remark that the Kronecker's Jugendtraum gives a strong motivation for studying absolute modular forms and absolute modular functions. It is a famous problem to construct abelian extensions (class fields) of an algebraic number field via division values of a suitable function. The studies in the rational number case (Kronecker) and the real quadratic field case (Shintani [S]) suggest looking at the extension $\mathbf{Q}\left(S_r\left(\frac{\omega_1+\cdots+\omega_r}{N},(\omega_1,\cdots,\omega_r)\right),\omega_1,\cdots,\omega_r\right)$ over $\mathbf{Q}(\omega_1,\cdots,\omega_r)$. This function $S_r\left(\frac{\omega_1+\cdots+\omega_r}{N},(\omega_1,\cdots,\omega_r)\right)$ on $\mathbf{D}_r$ (or the function $S_{r+1}\left(\frac{u_1+\cdots+u_r+1}{N},(u_1,\cdots,u_r,1)\right)$ on $\mathcal{D}_r$) is a typical absolute modular function (or an absolute modular form of weight 0).

In this paper we calculate $\mathscr{S}_k$ and $e_k$ to some extent in the one-variable case with a remark made on the two-variable case. The following four theorems concern the one-variable case.

**Theorem 1**

$$\mathscr{S}_1(u)=\frac{2\pi}{\sqrt{u}}.$$

**Theorem 2**

$$\mathscr{S}_2(u)=\begin{cases}\dfrac{8\pi^2 i}{\sqrt{u}}\left(\dfrac{1}{u}E_1\left(-\dfrac{1}{u}\right)-E_1(u)\right) & \textit{if } \mathrm{Im}(u)>0,\\[2ex] -\dfrac{8\pi^2 i}{\sqrt{u}}\left(\dfrac{1}{u}E_1\left(\dfrac{1}{u}\right)-E_1(-u)\right) & \textit{if } \mathrm{Im}(u)<0,\end{cases}$$

*where*

$$E_1(\tau)=-\frac{1}{4}+\sum_{n=1}^{\infty}d(n)e^{2\pi in\tau}$$

*for* $\mathrm{Im}(\tau)>0$ *with* $d(n)=\sum_{d|n}1$.

**Theorem 3**

$$e_0(u)=\frac{1}{12}\left(u+\frac{1}{u}-9\right).$$

**Theorem 4**

$$\mathscr{S}_3(u)=\frac{3}{4}\mathscr{S}_2(u)^2\mathscr{S}_1(u)^{-1}-\frac{3}{8}(4e_0(u)+3)\mathscr{S}_1(u)^3.$$

Among others, Theorem 2 shows an unexpected relation in weight 2 using to an Eisenstein-like series of weight 1; remark that $1/\sqrt{u}$ is of weight 1 as in Theorem 1. The identity in Theorem 4 is indicating a graded structure in weight 3. Our proof shows that this identity is coming from the (quasi-)modularity of

$$E_2(\tau)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma(n)e^{2\pi in\tau}$$

with

$$\sigma(n) = \sum_{d \mid n} d.$$

Here we are using the notation

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}$$

with

$$\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}.$$

In this paper we do not treat the "cuspidality." We think that the cuspidality of $\mathscr{S}_k(u)$ means that $\mathscr{S}_k(\infty) = 0$ at now. This vanishing is obvious for $\mathscr{S}_1(u)$ from Theorem 1. It is not so difficult to show $\mathscr{S}_2(\pm i\infty) = 0$ and $\mathscr{S}_3(\pm i\infty) = 0$ from Theorem 2 and Theorem 4, respectively. We hope to report on them in a future paper.

Our results have some applications. For example, from Theorem 2 and $\mathscr{S}_2(1) = -4\pi$ we see that:

**Corollary 1**

$$\lim_{\substack{\tau \to 1 \\ \mathrm{Im}(\tau) > 0}} \left( E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau) \right) = -\frac{1}{2\pi i}.$$

This indicates a merit of absolute modular forms, where the domain $\mathcal{D}_1$ contains the positive real numbers $\mathbf{R}_{>0}$ (See Figure 1). We refer to [K3] [KK3] for related results.

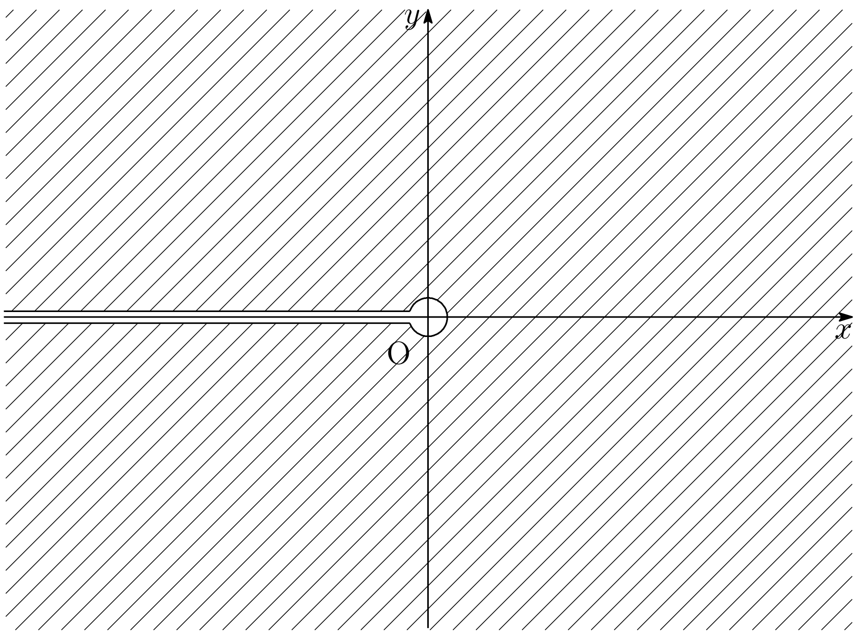


Figure 1. The domain $\mathcal{D}_1$

Lastly we notice calculations in the two-variable case:

**Theorem 5**

$$e_0(u,v) = \frac{1}{24}\left(u + \frac{1}{u} + v + \frac{1}{v} + \frac{u}{v} + \frac{v}{u} - 21\right).$$

**Theorem 6**

$$\begin{aligned}\mathscr{S}_1(u,v) &= \frac{\rho_3(u,v,1)^2\rho_1(u)\rho_1(v)\rho_1(1)}{\rho_2(u,v)\rho_2(u,1)\rho_2(v,1)} \\ &= \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{uv}} \cdot \frac{\rho_3(u,v,1)^2}{\rho_2(u,v)\rho_2(u,1)\rho_2(v,1)}.\end{aligned}$$

# 2 Proofs of Theorems 1 and 2

Since we have seen these results in other contexts essentially (see [K2] for example), we give concise proofs here.

*Proof of Theorem 1*:

It is sufficient to show that

$$S_2'(0,(\omega_1,\omega_2)) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}.$$

The periodicity proved in [KK1]

$$S_2(x,(\omega_1,\omega_2)) = S_2(x+\omega_2,(\omega_1,\omega_2))S_1(x,\omega_1)$$

with

$$S_1(x,\omega) = 2\sin\left(\frac{\pi x}{\omega}\right)$$

implies

$$S_2'(0,(\omega_1,\omega_2)) = S_2(\omega_2,(\omega_1,\omega_2))\frac{2\pi}{\omega_1}.$$

Here,

$$\begin{aligned}
S_2(\omega_2,(\omega_1,\omega_2)) &= \frac{\Gamma_2(\omega_1,(\omega_1,\omega_2))}{\Gamma_2(\omega_2,(\omega_1,\omega_2))} \\
&= \lim_{x\to 0}\frac{\Gamma_2(\omega_1+x,(\omega_1,\omega_2))}{\Gamma_2(\omega_2+x,(\omega_1,\omega_2))} \\
&= \lim_{x\to 0}\frac{\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_2)^{-1}}{\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}} \\
&= \lim_{x\to 0}\frac{\Gamma_1(x,\omega_1)}{\Gamma_1(x,\omega_2)} \\
&= \lim_{x\to 0}\frac{\frac{\Gamma(\frac{x}{\omega_1})}{\sqrt{2\pi}}\omega_1^{\frac{x}{\omega_1}-\frac{1}{2}}}{\frac{\Gamma(\frac{x}{\omega_2})}{\sqrt{2\pi}}\omega_2^{\frac{x}{\omega_2}-\frac{1}{2}}} \\
&= \sqrt{\frac{\omega_1}{\omega_2}}.
\end{aligned}$$

Thus

$$S_2'(0,(\omega_1,\omega_2)) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}.$$

∎

*Proof of Theorem 2*:

A formula of Shintani [S] (Proposition 5) shows that

$$\begin{aligned}
\mathrm{Cot}_2(x,(1,\tau)) &= \pi i\left(\frac{x}{\tau}-\frac{1}{2}\left(1+\frac{1}{\tau}\right)\right) \\
&\quad -2\pi i\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}e^{2\pi i n m\tau}e^{2\pi i m x}+\frac{2\pi i}{\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{-\frac{2\pi i n m}{\tau}}e^{\frac{2\pi i m x}{\tau}},
\end{aligned}$$

where we use the double cotangent function

$$\mathrm{Cot}_2(x,(1,u)) = \frac{S_2'(x,(1,u))}{S_2(x,(1,u))}.$$

Hence

$$\begin{aligned}
\mathrm{Cot}_2(\tau,(1,\tau)) &= \frac{\pi i}{2}\left(1-\frac{1}{\tau}\right)-2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{2\pi i n m\tau}+\frac{2\pi i}{\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{-\frac{2\pi i n m}{\tau}} \\
&= \frac{\pi i}{2}\left(1-\frac{1}{\tau}\right)-2\pi i\left(E_1(\tau)+\frac{1}{4}\right)+\frac{2\pi i}{\tau}\left(E_1\left(-\frac{1}{\tau}\right)+\frac{1}{4}\right) \\
&= \frac{2\pi i}{\tau}\left(E_1\left(-\frac{1}{\tau}\right)-\tau E_1(\tau)\right).
\end{aligned}$$

Now we prove that
$$S_2''(0,(1,\tau)) = \frac{4\pi}{\sqrt{\tau}}\mathrm{Cot}_2(\tau,(1,\tau)).$$
This is obtained from the periodicity
$$S_2(x,(1,\tau)) = S_2(x+\tau,(1,\tau))S_1(x)$$
with $S_1(x) = 2\sin(\pi x)$. In fact, differentiating this periodicity relation twice leads to
$$S_2''(x,(1,\tau)) = S_2''(x+\tau,(1,\tau))S_1(x) + 2S_2'(x+\tau,(1,\tau))S_1'(x) + S_2(x+\tau,(1,\tau))S_1''(x).$$
Hence, we obtain the desired relation
$$\begin{aligned} S_2''(0,(1,\tau)) &= 2S_2'(\tau,(1,\tau))S_1'(0) \\ &= 4\pi S_2'(\tau,(1,\tau)) \\ &= \frac{4\pi}{\sqrt{\tau}} \cdot \frac{S_2'(\tau,(1,\tau))}{S_2(\tau,(1,\tau))} \\ &= \frac{4\pi}{\sqrt{\tau}}\mathrm{Cot}_2(\tau,(1,\tau)), \end{aligned}$$
where we used the fact
$$S_2(\tau,(1,\tau)) = \frac{1}{\sqrt{\tau}}$$
proved in [KK2]. Thus we have
$$\mathscr{S}_2(\tau) = S_2''(0,(1,\tau)) = \frac{8\pi^2 i}{\tau\sqrt{\tau}}\left(E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau)\right).$$
This proves Theorem 2 for $\mathrm{Im}(u) > 0$. The case $\mathrm{Im}(u) < 0$ is given by the reflection. ∎

# 3 Proof of Theorem 3

We prove that
$$\zeta_2(0,(\omega_1,\omega_2)) = \frac{1}{12}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 9\right).$$
For this we recall the Riemann-Mellin type integral expression for $\zeta_2(s,(\omega_1,\omega_2))$ (cf. Barnes [B]). It says that
$$\zeta_2(s,(\omega_1,\omega_2)) = \frac{1}{\Gamma(s)}\int_0^\infty \Theta(t)t^{s-1}dt$$

in $\mathrm{Re}(s) > 2$ with

$$\begin{aligned}\Theta(t) &= \sideset{}{'}\sum_{n_1, n_2 \geq 0} e^{-(n_1\omega_1 + n_2\omega_2)t} \\ &= \frac{1}{(1 - e^{-t\omega_1})(1 - e^{-t\omega_2})} - 1 \\ &= \frac{e^{t\omega_1} + e^{t\omega_2} - 1}{(e^{t\omega_1} - 1)(e^{t\omega_2} - 1)}.\end{aligned}$$

Let

$$\Theta(t) = \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + a_1 t + \cdots$$

be the Laurent expansion around $t = 0$. An easy calculation shows that

$$a_{-2} = \frac{1}{\omega_1\omega_2}, \qquad a_{-1} = \frac{1}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right), \qquad a_0 = \frac{1}{12}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 9\right).$$

Now, to make an analytic continuation of $\zeta_2(s, (\omega_1, \omega_2))$ in $\mathrm{Re}(s) > -1$ we split the integral into three parts:

$$\begin{aligned}\zeta_2(s, (\omega_1, \omega_2)) &= \frac{1}{\Gamma(s)} \int_1^\infty \Theta(t) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_0^1 \left(\Theta(t) - \frac{a_{-2}}{t^2} - \frac{a_{-1}}{t} - a_0\right) t^{s-1} dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^1 \left(\frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0\right) t^{s-1} dt.\end{aligned}$$

Here, the first term is holomorphic for all $s \in \mathbf{C}$ since the integral is absolutely convergent. The second term is holomorphic in $\mathrm{Re}(s) > -1$ since

$$\Theta(t) - \frac{a_{-2}}{t^2} - \frac{a_{-1}}{t} - a_0 = O(t)$$

as $t \to 0$. The third term is

$$\frac{1}{\Gamma(s)}\left(\frac{a_{-2}}{s-2} + \frac{a_{-1}}{s-1} + \frac{a_0}{s}\right),$$

which is meromorphic in $s \in \mathbf{C}$. Hence, the above expression gives an analytic continuation of $\zeta_2(s, (\omega_1, \omega_2))$ in $\mathrm{Re}(s) > -1$. Moreover this calculation shows that $\zeta_2(s, (\omega_1, \omega_2))$ is holomorphic at $s = 0$ and its value is given by $\zeta_2(0, (\omega_1, \omega_2)) = a_0$. Thus

$$\zeta_2(0, (\omega_1, \omega_2)) = \frac{1}{12}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 9\right).$$

∎

# 4 Proof of Theorem 4

We prove the following facts (a)(b):

(a)

$$S_2'''(0,(1,\tau)) = \frac{6\pi}{\sqrt{\tau}}\left(\mathrm{Cot}_2'(\tau,(1,\tau)) + \mathrm{Cot}_2(\tau,(1,\tau))^2 - \frac{\pi^2}{3}\right),$$

(b)

$$\mathrm{Cot}_2'(\tau,(1,\tau)) = \frac{\pi^2}{6}\left(1 - \frac{1}{\tau^2}\right).$$

Before proving (a) and (b) we show that Theorem 4 follows from them. In fact (a) says that

$$\begin{aligned}\mathscr{S}_3(\tau) &= S_2'''(0,(1,\tau)) \\ &= \frac{6\pi}{\sqrt{\tau}}\mathrm{Cot}_2'(\tau,(1,\tau)) + \frac{6\pi}{\sqrt{\tau}}\mathrm{Cot}_2(\tau,(1,\tau))^2 - \frac{2\pi^3}{\sqrt{\tau}},\end{aligned}$$

so using (b) and the formula

$$\begin{aligned}\mathrm{Cot}_2(\tau,(1,\tau)) &= \frac{\sqrt{\tau}}{4\pi}S_2''(0,(1,\tau)) \\ &= \frac{\sqrt{\tau}}{4\pi}\mathscr{S}_2(\tau)\end{aligned}$$

proved in §2, we have

$$\begin{aligned}\mathscr{S}_3(\tau) &= \frac{\pi^3}{\sqrt{\tau}}\left(1 - \frac{1}{\tau^2}\right) + \frac{3\sqrt{\tau}}{8\pi}\mathscr{S}_2(\tau)^2 - \frac{2\pi^3}{\sqrt{\tau}} \\ &= -\frac{1}{8}\left(\frac{2\pi}{\sqrt{\tau}}\right)^3\left(\tau + \frac{1}{\tau}\right) + \frac{3}{4}\left(\frac{2\pi}{\sqrt{\tau}}\right)^{-1}\mathscr{S}_2(\tau)^2 \\ &= -\frac{1}{8}\mathscr{S}_1(\tau)^3(12e_0(\tau)+9) + \frac{3}{4}\mathscr{S}_1(\tau)^{-1}\mathscr{S}_2(\tau)^2.\end{aligned}$$

This is Theorem 4.

*Proof of (a)*:
The definition

$$\mathrm{Cot}_2(x,(1,\tau)) = \frac{S_2'(x,(1,\tau))}{S_2(x,(1,\tau))}$$

implies

$$\mathrm{Cot}_2'(x,(1,\tau)) = \frac{S_2''(x,(1,\tau))}{S_2(x,(1,\tau))} - \frac{S_2'(x,(1,\tau))^2}{S_2(x,(1,\tau))^2}.$$

Hence

$$\begin{aligned}\mathrm{Cot}_2'(\tau,(1,\tau)) &= \frac{S_2''(\tau,(1,\tau))}{S_2(\tau,(1,\tau))} - \left(\frac{S_2'(\tau,(1,\tau))}{S_2(\tau,(1,\tau))}\right)^2 \\ &= \frac{S_2''(\tau,(1,\tau))}{S_2(\tau,(1,\tau))} - \mathrm{Cot}_2(\tau,(1,\tau))^2.\end{aligned}$$

Now we show that

$$S_2''(\tau,(1,\tau)) = \frac{1}{6\pi}\left(S_2'''(0,(1,\tau)) + \frac{2\pi^3}{\sqrt{\tau}}\right).$$

We obtain this by differentiating the periodicity relation

$$S_2(x,(1,\tau)) = S_2(x+\tau,(1,\tau))S_1(x)$$

three times at $x=0$. In fact the identity

$$\begin{aligned}S_2'''(x,(1,\tau)) &= S_2'''(x+\tau,(1,\tau))S_1(x) + 3S_2''(x+\tau,(1,\tau))S_1'(x) \\ &\quad +3S_2'(x+\tau,(1,\tau))S_1''(x) + S_2(x+\tau,(1,\tau))S_1'''(x)\end{aligned}$$

gives

$$\begin{aligned}S_2'''(0,(1,\tau)) &= 6\pi S_2''(\tau,(1,\tau)) - 2\pi^3 S_2(\tau,(1,\tau)) \\ &= 6\pi S_2''(\tau,(1,\tau)) - \frac{2\pi^3}{\sqrt{\tau}}\end{aligned}$$

since $S_1(x) = 2\sin(\pi x)$ and $S_2(\tau,(1,\tau)) = \frac{1}{\sqrt{\tau}}$. Thus

$$\mathrm{Cot}_2'(\tau,(1,\tau)) = \frac{\sqrt{\tau}}{6\pi}S_2'''(0,(1,\tau)) + \frac{\pi^2}{3} - \mathrm{Cot}_2(\tau,(1,\tau))^2.$$

This gives (a).

*Proof of (b)*:

We use the method of the proof of Theorem 2. In this case we start from

$$\mathrm{Cot}_2'(x,(1,\tau)) = \frac{\pi i}{\tau} + 4\pi^2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m e^{2\pi i n m\tau}e^{2\pi i m x} - \frac{4\pi^2}{\tau^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m e^{-\frac{2\pi i n m}{\tau}}e^{\frac{2\pi i m x}{\tau}}.$$

Then we have

$$\begin{aligned}\mathrm{Cot}_2'(\tau,(1,\tau)) &= \frac{\pi i}{\tau} + 4\pi^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m e^{2\pi i n m\tau} - \frac{4\pi^2}{\tau^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m e^{-\frac{2\pi i n m}{\tau}} \\ &= \frac{\pi i}{\tau} + 4\pi^2\left(E_2(\tau) + \frac{1}{24}\right) - \frac{4\pi^2}{\tau^2}\left(E_2\left(-\frac{1}{\tau}\right) + \frac{1}{24}\right) \\ &= 4\pi^2\left(E_2(\tau) - \frac{1}{\tau^2}E_2\left(-\frac{1}{\tau}\right)\right) + \frac{\pi i}{\tau} + \frac{\pi^2}{6}\left(1-\frac{1}{\tau^2}\right) \\ &= \frac{\pi^2}{6}\left(1-\frac{1}{\tau^2}\right),\end{aligned}$$

where we used the modularity of $E_2(\tau)$:

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{\tau}{4\pi i}.$$

This proves (b). ∎

**Remark** Similar calculations show that

$$\mathscr{S}_5 = \frac{5}{2}\mathscr{S}_4\mathscr{S}_2\mathscr{S}_1^{-1} - \frac{15}{16}\mathscr{S}_2^4\mathscr{S}_1^{-3} + \frac{15}{16}(4e_0+3)\mathscr{S}_2^2\mathscr{S}_1 + \frac{1}{192}(144e_0^2+216e_0+89)\mathscr{S}_1^5.$$

# 5 Proof of Corollary 1

Theorem 2 implies

$$\begin{aligned}\lim_{\substack{\tau\to 1\\ \mathrm{Im}(\tau)>0}}\left(E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau)\right) &= \frac{1}{8\pi^2 i}\mathscr{S}_2(1)\\ &= \frac{1}{8\pi^2 i}S_2''(0,(1,1)),\end{aligned}$$

and Theorem 4(1) of [K2] says that

$$S_2''(0,(1,1)) = -4\pi.$$

Hence we get Corollary 1. ∎

# 6 Proof of Theorem 5

We calculate $\zeta_3(0,(\omega_1,\omega_2,\omega_3))$. Exactly similarly to the case of $\zeta_2(0,(\omega_1,\omega_2))$ treated in §3 we have

$$\zeta_3(0,(\omega_1,\omega_2,\omega_3)) = a_0 = a_0(\omega_1,\omega_2,\omega_3)$$

where $a_0$ is the constant term of the Laurent expansion of

$$\Theta(t) = \frac{1}{(1-e^{-t\omega_1})(1-e^{-t\omega_2})(1-e^{-t\omega_3})} - 1$$

around $t=0$:

$$\Theta(t) = \frac{a_{-3}}{t^3} + \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + a_1 t + \cdots.$$

The direct calculation shows that

$$a_0 = \frac{\omega_1\omega_2^2+\omega_2\omega_3^2+\omega_3\omega_1^2+\omega_1^2\omega_2+\omega_2^2\omega_3+\omega_3^2\omega_1-21\omega_1\omega_2\omega_3}{24\omega_1\omega_2\omega_3}.$$

Hence

$$\begin{aligned} e_0(u,v) &= \zeta_3(0,(u,v,1)) \\ &= \frac{1}{24}\left(u+\frac{1}{u}+v+\frac{1}{v}+\frac{u}{v}+\frac{v}{u}-21\right). \end{aligned}$$

∎

# 7 Proof of Theorem 6

We calculated $S_3'(0,(\omega_1,\omega_2,\omega_3))$ in [K2] as

$$S_3'(0,(\omega_1,\omega_2,\omega_3)) = \frac{\rho_3(\omega_1,\omega_2,\omega_3)^2\rho_1(\omega_1)\rho_2(\omega_2)\rho_3(\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_3,\omega_1)}$$

with

$$\rho_1(\omega) = \sqrt{\frac{2\pi}{\omega}}.$$

Hence

$$\begin{aligned} \mathscr{S}_1(u,v) &= S_3'(0,(u,v,1)) \\ &= \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{uv}} \cdot \frac{\rho_3(u,v,1)^2}{\rho_2(u,v)\rho_2(u,1)\rho_2(v,1)}. \end{aligned}$$

∎

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