Refinement of prime geodesic theorem

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Abstract

We prove existence of a set E of positive real numbers, which is relatively small in the sense that its logarithmic measure is finite, such that we can improve the error term of the prime geodesic theorem as $x \to \infty$ $(x \notin E)$. The result holds for any compact hyperbolic surfaces, and it would also be true for generic hyperbolic surfaces of finite volume according to the philosophy of Phillips and Sarnak.

1 Introduction

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a cofinite Fuchsian group. Denote

$$\psi_\Gamma(x) = \sum_{\substack{p,k \\ N(p)^k \leq x}} \log N(p),$$

where p runs through primitive hyperbolic conjugacy classes (i.e. prime geodesics on $\Gamma \backslash H$ with H the upper half plane), $N(p) = \exp(\text{length}(p))$, and k runs through positive integers.

Prime geodesic theorem asserts that

$$\psi_\Gamma(x) \sim x \qquad (x \to \infty).$$

Our chief concern is the error term of this formula. We now briefly explain how it depends on the estimate of an exponential sum over zeros of the Selberg zeta function.

For simplicity we start from the modular surface case. When $\Gamma = PGL(2, \mathbb{Z})$, an analog of the Riemann hypothesis for the Selberg zeta function $Z_\Gamma(s)$ is known to hold. That is, all nontrivial zeros of $Z_\Gamma(s)$

lie on the line $\mathrm{Re}(s) = 1/2$. By [I] Lemma 1, it holds for $1 \leq T \leq \sqrt{x}(\log x)^{-2}$ that

$$\psi_\Gamma(x) = x + \sum_{|\gamma_j| \leq T} \frac{x^{\rho_j}}{\rho_j} + O\left(\frac{x}{T}(\log x)^2\right) \qquad (T \to \infty), \tag{1}$$

where $\rho_j = \frac{1}{2} + i\gamma_j$ runs over the zeros of the Selberg zeta function $Z_\Gamma(s)$ counted with multiplicity. It is known that

$$\#\{j : \ |\gamma_j| \leq T\} = \frac{T^2}{12} + c_1 T \log T + O(T) \qquad (T \to \infty)$$

for some constant c_1. Therefore by estimating the sum over γ_j in (1) trivially, it holds that

$$\psi_\Gamma(x) = x + O\left(x^{\frac{1}{2}} T + \frac{x}{T}(\log x)^2\right) \qquad (T \to \infty).$$

On taking the optimal value $T = x^{1/4} \log x$, we have

$$\psi_\Gamma(x) = x + O\left(x^{\frac{3}{4}} \log x\right) \qquad (x \to \infty).$$

For more general cofinite Γ, the formula (1) is not explicitly proved, but the estimate of the sum

$$\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ |\mathrm{Im}(\rho)|<T}} \frac{x^{i\gamma}}{\rho}$$

still controls the error term. By a detailed analysis of the Selberg trace formula, a little better estimate for the error term is known as

$$O\left(x^{\frac{3}{4}} (\log x)^{\frac{1}{2}}\right) \qquad (T \to \infty) \tag{2}$$

in [H1] Theorem 6.18 (p.111) and [H2] Theorem 3.4 (p.474) for cocompact and cofinite Γ, respectively.

In the past research the error terms in the prime geodesic theorem have been successfully improved only for arithmetic special cases. Iwaniec [I] obtained $O(x^{\frac{35}{48}+\varepsilon})$ for $\Gamma = PSL(2, \mathbb{Z})$, Luo-Sarnak [LS] improved it to $O(x^{\frac{7}{10}+\varepsilon})$ and Luo-Rudnick-Sarnak [LRS] generalized it for any congruence subgroups in $PSL(2, \mathbb{Z})$. Then the author [K] extended it to arithmetic cocompact groups.

Thus the situation is quite different from the classical case on prime number theorem, where we have

$$\psi(x) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right) \qquad (x \to \infty) \tag{3}$$

under assuming the Riemann Hypothesis. This difference is caused by the abundance of zeros for $Z_\Gamma(s)$. The true size of the error term in the prime geodesic theorem is unknown. If the distribution of Laplace eigenvalues is uniform enough to make a huge cancellation occur, then the error term may be as good as $O(x^{\frac{1}{2}+\varepsilon})$. If this is correct, we would say that we are still standing very far from the truth, even if we almost proved the Riemann Hypothesis.

According to Hejhal [H1] Theorem 15.13 (p.252), the error term for the prime geodesic theorem is expressed for cocompact cases as

$$O\left(x^{\frac{1+2\alpha}{2+2\alpha}}(\log x)^{\frac{1}{1+\alpha}}\right) \qquad (x \to \infty),$$

if $S(t) = \frac{1}{\pi}\arg Z_\Gamma(\frac{1}{2}+it)$ satisfies that $S(t) = O(t^\alpha)$ for some $0 < \alpha < 1$. Taking $\alpha \to 1$ would recover the trivial bound above. If $S(t)$ were bounded, we would take $\alpha \to 0$ to get $O(x^{\frac{1}{2}}\log x)$. But we do not have any evidence for that.

In this paper we obtain an evidence for the trivial error term (2) is not optimal. For describing our result, we define the logarithmic measure $\mu^\times(E)$ of a measurable subset $E \subset \mathbb{R}_{\geq 2}$ as

$$\mu^\times(E) = \int_E \frac{dx}{x}.$$

When Γ is cofinite but not cocompact, there exist continuous spectra and the generalized Weyl law asserts that

$$\begin{aligned}\#\{j:\ |\gamma_j| \leq T\} &+ \frac{1}{4\pi}\int_{-T}^{T}\frac{\varphi'(\frac{1}{2}+it)}{\varphi(\frac{1}{2}+it)}dt \\ &= \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}T^2 + c_1 T\log T + O(T) \qquad (T \to \infty)\end{aligned} \tag{4}$$

for some constant c_1 with φ the scattering determinant. We call Γ *ordinary* if it satisfies either of the following two conditions:

(i) The first term is the main term in the left hand side of (4). That is

$$\#\{j:\ |\gamma_j| \leq T\} \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}T^2 \qquad (T \to \infty).$$

(ii) The second term is the main term, and the first term satisfies that

$$\#\{j : |\gamma_j| \le T\} = O(T) \qquad (T \to \infty).$$

Any congruence subgroup Γ satisfies the condition (i). On the other hand, a generic cofinite Γ should satisfy (ii) according to the philosophy of Phillips and Sarnak [PS].

Theorem 1. *For any cocompact or ordinary cofinite* Γ*, there exists a subset* E *in* $\mathbb{R}_{\ge 2}$ *whose logarithmic measure is finite, such that*

$$\psi_\Gamma(x) = x + \sum_{\mathrm{Re}(\rho) > \frac{1}{2}} \frac{x^\rho}{\rho} + O\left(x^{\frac{3}{4}} (\log\log x)^{\frac{1}{4}+\varepsilon}\right) \qquad (x \to \infty,\ x \notin E),$$

where ρ *runs through the finite number of exceptional zeros of the Selberg zeta function.*

Theorem 1 gives an improvement of (2). The proof is analogous to that of Gallagher [G2] for the Riemann zeta function. His result was conditional in the sense that he had to assume the Riemann Hypothesis. In our case, however, we do not need any assumption for proving the existence of the set E, because the Riemann Hypothesis is proved for Selberg zeta functions except for at most finitely many exceptional zeros.

2 Proofs

We quote a lemma discovered earlier by Gallagher [G2].

Lemma 1 (Gallagher[G1])**.** *Let* A *be a discrete subset of* $\mathbb{R}$*. For any given sequence* $c(\nu) \in \mathbb{C}$ $(\nu \in A)$*, assume that the series*

$$S(u) = \sum_{\nu \in A} c(\nu) e^{2\pi i \nu u}$$

is absolutely convergent. Then for any $\theta \in (0, 1)$*, it holds that*

$$\int_{-U}^{U} |S(u)|^2 du \le \left(\frac{\pi\theta}{\sin(\pi\theta)}\right)^2 \int_{-\infty}^{\infty} \left| \frac{U}{\theta} \sum_{t \le \nu \le t + \frac{\theta}{U}} c(\nu) \right|^2 dt.$$

Let

$$\psi_{1,\Gamma}(x) = \int_0^x \psi_\Gamma(u)du.$$

Theorem 2. *For any cocompact or ordinary cofinite* Γ*, there exists a subset* E *in* $\mathbb{R}$ *whose logarithmic measure is finite, such that for any* $\varepsilon > 0$ *it holds that*

$$\psi_{1,\Gamma}(x) = \frac{x^2}{2} + \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2} \\ |\gamma|\le X}} \frac{x^{\rho+1}}{\rho(\rho+1)} + O\left(x^{\frac{3}{2}}(\log\log x)^{\frac{1}{2}+\varepsilon}\right) \qquad (x\to\infty,\ x\notin E).$$

Proof. By Hejhal [H1] Theorem 6.16 (p.110), it holds that

$$\begin{aligned}\psi_{1,\Gamma}(x) &= \alpha_0 x + \beta_0 x\log x + \alpha_1 + \beta_1 \log x + F\left(\frac{1}{x}\right) \\ &+ \frac{x^2}{2} + \sum_{\substack{\rho \\ |\mathrm{Im}(\rho)|<T}} \frac{x^{1+\rho}}{\rho(1+\rho)} + O\left(\frac{x^2\log x}{T}\right) \qquad (T, x\to\infty) \quad (5)\end{aligned}$$

with some constants α_0, α_1, β_0, β_1 and

$$F(x) = (2g-2)\sum_{k=2}^{\infty} \frac{2k+1}{k(k-1)} x^{k-1}.$$

Actually the above theorem is explicitly proved for cocompact Γ, and it is generalized by himself in [H2] in the proof of Theorem 3.4 (p.474) for cofinite Γ.

Since

$$\sum_{\substack{\rho \\ |\gamma|\le X}} \frac{x^{\rho+1}}{\rho(\rho+1)} = \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2} \\ |\gamma|\le X}} \frac{x^{\rho+1}}{\rho(\rho+1)} + \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ |\gamma|\le X}} \frac{x^{\rho+1}}{\rho(\rho+1)},$$

where the first term in the left hand side is a finite sum, it suffices to estimate the last sum

$$\sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ |\gamma|\le X}} \frac{x^{\rho+1}}{\rho(\rho+1)}.$$

For $Y \leq X$, we have by putting $x = Xe^{2\pi u}$ that

$$\int_X^{eX} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dx}{x^4} = \int_X^{eX} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{x^{i\gamma}}{\rho(\rho+1)} \right|^2 \frac{dx}{x}$$

$$= 2\pi \int_0^{1/2\pi} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{X^{i\gamma}}{\rho(\rho+1)} e^{2\pi i\gamma u} \right|^2 du$$

$$= 2\pi \int_{-1/4\pi}^{1/4\pi} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{X^{i\gamma}}{\rho(\rho+1)} e^{2\pi i\gamma(u+\frac{1}{4\pi})} \right|^2 du.$$

We apply Lemma 1 with $\nu = \gamma = \mathrm{Im}(\rho)$, $\theta = U = 1/4\pi$ and

$$c(\gamma) = \begin{cases} \frac{X^{i\gamma}}{\rho(\rho+1)} e^{i\gamma/2} & (Y < |\gamma| \leq X) \\ 0 & (\text{otherwise}) \end{cases}.$$

Thus

$$\int_X^{eX} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dx}{x^4} \leq \left(\frac{\frac{1}{4}}{\sin\frac{1}{4}} \right)^2 \int_{-\infty}^{\infty} \left(\sum_{\substack{t<|\rho|\leq t+1 \\ Y<|\gamma|\leq X}} \frac{1}{|\rho(\rho+1)|} \right)^2 dt.$$

The number of γ participating in the sum is $O(t)$ by Weyl's law:

$$\#\{j :\ |\gamma_j| \leq T\} = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2 + c_1 T \log T + O(T) \qquad (T \to \infty)$$

for cocompact Γ, and by the assumption that Γ is ordinary for cofinite Γ. Since $\frac{1}{\rho(\rho+1)} = O(t^{-2})$ $(t \to \infty)$, we have

$$\int_X^{eX} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\leq X}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dx}{x^4} = O\left(\int_{Y-1}^{X+1} \frac{1}{t^2} dt \right)$$

$$= O(Y^{-1}) \qquad (Y \to \infty).$$

In particular, when $X = e^Y$,

$$\int_{e^Y}^{e^{Y+1}} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\le e^Y}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dx}{x^4} = O(Y^{-1}).$$

Define a set E_Y by

$$E_Y = \left\{ x \in \left[e^Y, e^{Y+1}\right] \ : \ \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\le e^Y}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| \ge x^{\frac{3}{2}} (\log\log x)^{\frac{1}{2}+\varepsilon} \right\}.$$

We denote its logarithmic measure by $M_Y = \mu^\times(E_Y)$.

$$\begin{aligned} Y^{-1} \gg \int_{E_Y} \left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\le e^Y}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dx}{x^4} &\ge \int_{E_Y} \left(x^{\frac{3}{2}} (\log\log x)^{\frac{1}{2}+\varepsilon} \right)^2 \frac{dx}{x^4} \\ &\ge \left(\log\log e^Y\right)^{1+2\varepsilon} \int_{E_Y} \frac{dx}{x} \\ &= (\log Y)^{1+2\varepsilon} M_Y. \end{aligned}$$

Hence

$$M_Y = O\left(\frac{1}{Y(\log Y)^{1+2\varepsilon}} \right).$$

Then the logarithmic measure of the set

$$E = \bigcup_{Y=2}^{\infty} E_Y$$

is estimated by

$$M \ll \sum_{Y=2}^{\infty} M_Y \le \sum_{Y=2}^{\infty} \frac{1}{Y(\log Y)^{1+2\varepsilon}} \ll \int_2^\infty \frac{1}{x(\log x)^{1+2\varepsilon}} dx < \infty.$$

By the definition of E, it holds for any $x \in \mathbb{R} \setminus E$ that

$$\left| \sum_{\substack{\mathrm{Re}(\rho)=\frac{1}{2} \\ Y<|\gamma|\le e^Y}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| < \sqrt{x}^3 (\log\log x)^{\frac{1}{2}+\varepsilon}.$$

□

Proof of Theorem 1. Since ψ is an increasing function, we have

$$\begin{aligned}\psi(x) &\leq \frac{1}{h}\int_x^{x+h}\psi(t)dt\\ &= \frac{1}{h}\left(\psi_1(x+h)-\psi_1(x)\right).\end{aligned}$$

Thus if we denote the error term by $R(x)$ as

$$\psi(x) = x + \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2}\\ |\gamma|\leq X}}\frac{x^\rho}{\rho} + R(x),$$

it holds from Theorem 2 that

$$\begin{aligned}|R(x)| &= \left|\psi(x) - x - \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2}\\ |\gamma|\leq X}}\frac{x^\rho}{\rho}\right|\\ &\leq \left|\frac{\psi_1(x+h)-\psi_1(x)}{h} - x - \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2}\\ |\gamma|\leq X}}\frac{x^\rho}{\rho}\right|\\ &= \left|\frac{1}{h}\left(\frac{(x+h)^2}{2} - \frac{x^2}{2} - \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2}\\ |\gamma|\leq X}}\frac{(x+h)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)} + O\left(x^{\frac{3}{2}}(\log\log x)^{\frac{1}{2}+\varepsilon}\right)\right)\right.\\ &\qquad\left. - x - \sum_{\substack{\mathrm{Re}(\rho)>\frac{1}{2}\\ |\gamma|\leq X}}\frac{x^\rho}{\rho}\right| \quad (x\to\infty,\ x\notin E)\\ &= \left|\frac{h}{2} + O\left(\frac{x^{\frac{3}{2}}(\log\log x)^{\frac{1}{2}+\varepsilon}}{h}\right)\right| \qquad (x\to\infty,\ x\notin E).\end{aligned}$$

By choosing

$$h = x^{\frac{3}{4}}(\log\log x)^{\frac{1}{4}+\varepsilon},$$

we have

$$R(x) = O\left(x^{\frac{3}{4}}(\log\log x)^{\frac{1}{4}+\varepsilon}\right) \qquad (x \to \infty,\ x \notin E).$$

$\square$

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