Zeta Functions and Normalized Multiple Sine Functions

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Abstract. By using normalized multiple sine functions we show expressions for special values of zeta functions and L-functions containing $\zeta(3)$, $\zeta(5)$, etc. Our result reveals the importance of division values of normalized multiple sine functions. Properties of multiple Hurwitz zeta functions are crucial for the proof.

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1 Introduction

The normalized sine function

$$S_1(x) = 2\sin(\pi x)$$

has the basic importance in number theory. This is expressed as

$$S_1(x) = \Gamma_1(x)^{-1}\Gamma_1(1-x)^{-1}$$

with the normalized gamma function

$$\Gamma_1(x) = \exp\left(\frac{\partial}{\partial s}\zeta(s,x)\Big|_{s=0}\right),$$

where

$$\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$$

is the Hurwitz zeta function. In fact, Lerch's formula says that

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}$$

for the usual gamma function $\Gamma(x)$.

We know that the special value of $S_1(x)$ at a rational number $x \in \mathbb{Q}$ with $0 < x < 1$ is an algebraic integer

$$\begin{aligned} S_1(x) &= \left(e^{-\pi i x} - e^{\pi i x}\right) i \\ &= \left|1 - e^{2\pi i x}\right|. \end{aligned}$$

(Since $S_1(1/3) = \sqrt{3}$, the factor "2" is needed in $S_1(x)$ to assure the integrality.) This algebraic integer is intimately related to the cyclotomic unit, and at the same time it appears in the socalled the class number formula of Dirichlet

$$L(1, \chi) = -\frac{\tau(\chi)}{N} \log \left(\prod_{k=1}^{N-1} S_1 \left(\frac{k}{N} \right)^{\bar{\chi}(k)} \right),$$

where

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

is the Dirichlet L-function for a non-trivial primitive even Dirichlet character χ modulo N and

$$\tau(\chi) = \sum_{k=1}^{N-1} \chi(k) e^{2\pi i k/N}$$

is the Gauss sum. We notice that the Dirichlet's formula is written also as

$$L'(0, \chi) = -\frac{1}{2} \log \left(\prod_{k=1}^{N-1} S_1 \left(\frac{k}{N} \right)^{\chi(k)} \right)$$

via the functional equation.

The purpose of this paper is to generalize such a formula to the case of $L(r, \chi)$ for $r \geq 2$ containing the Riemann zeta case $\chi = \mathbf{1}$ by using the normalized multiple sine function $S_r(x)$, which was constructed and studied in the previous paper [KK] (see §2 for a survey). We recall the construction. For $\omega_1, ..., \omega_r > 0$ and $x > 0$, the multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, x; (\omega_1, ..., \omega_r)) = \sum_{n_1, ..., n_r = 0}^{\infty} (n_1 \omega_1 + \cdots + n_r \omega_r + x)^{-s}$$

in $\mathrm{Re}(s) > r$. This has the analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function, and it is holomorphic at $s = 0$. Then the normalized multiple gamma function is defined as

$$\Gamma_r(x, (\omega_1, ..., \omega_r)) = \exp \left(\frac{\partial}{\partial s} \zeta_r(s, x; (\omega_1, ..., \omega_r)) \Big|_{s=0} \right).$$

This is a constant multiple of multiple gamma function $\Gamma_r^B(x;(\omega_1,...,\omega_r))$ of Barnes [B]:

$$\Gamma_r(x;(\omega_1,...,\omega_r)) = \Gamma_r^B(x;(\omega_1,...,\omega_r))/\rho_r(\omega_1,...,\omega_r).$$

Now, the normalized multiple sine function is

$$S_r(x;(\omega_1,...,\omega_r)) = \Gamma_r(x;(\omega_1,...,\omega_r))^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-x;(\omega_1,...,\omega_r))^{(-1)^r}.$$

Hence, by the zeta regularized product (see Manin [M]), we can write

$$S_r(x;(\omega_1,...,\omega_r)) = \prod_{n_1,\ldots,n_r=0}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r+x)\left(\prod_{n_1,\ldots,n_r=1}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r-x)\right)^{(-1)^{r-1}}.$$

For example

$$\begin{aligned} S_1(x,\omega) &= \Gamma_1(x,\omega)^{-1}\Gamma_1(\omega-x,\omega)^{-1} \\ &= \prod_{n=0}^{\infty}(n\omega+x)\prod_{n=1}^{\infty}(n\omega-x) \\ &= 2\sin(\pi x/\omega) \end{aligned}$$

since we have

$$\Gamma_1(x,\omega) = (2\pi)^{-1/2}\Gamma(x/\omega)\omega^{\frac{x}{\omega}-\frac{1}{2}}$$

from

$$\zeta_1(s,x,\omega) = \omega^{-s}\zeta(s,x/\omega).$$

To simplify the notation we put $S_r(x) = S_r(x;(1,...,1))$, $\Gamma_r(x) = \Gamma_r(x;(1,...,1))$ and $\zeta_r(s,x) = \zeta_r(s,x;(1,...,1))$. Hence

$$S_r(x) = \Gamma_r(x)^{-1}\Gamma_r(r-x)^{(-1)^r}$$

and

$$\Gamma_r(x) = \exp\left(\left.\frac{\partial}{\partial s}\zeta_r(s,x)\right|_{s=0}\right).$$

This normalized multiple sine function $S_r(x)$ has good properties similar to the usual sine function $S_1(x) = 2\sin(\pi x)$. We refer to §2 for details. For example, it has the periodicity and the duplication formula:

$$S_r(x+1) = S_r(x)S_{r-1}(x)^{-1}$$

and

$$S_r(2x) = \prod_{k=0}^{r} S_r\left(x+\frac{k}{2}\right)^{\binom{r}{k}}.$$

Moreover $S_r(x)$ satisfies the following differential equation:

$$\frac{S_r'(x)}{S_r(x)} = Q_r(x)\cot \pi x$$

with $Q_r(x) = (-1)^{r-1}\pi\binom{x-1}{r-1}$. So, $S_r(x)$ is a solution of the algebraic differential equation

$$S_r''(x) + \left(\pi Q_r(x)^{-1} - 1\right) S_r'(x)^2 S_r(x)^{-1} - Q_r'(x)Q_r(x)^{-1}S_r'(x) + \pi Q_r(x)S_r(x) = 0.$$

We also note that $S_r(x)$ has the Weierstrass product expression similar to

$$\begin{aligned} S_1(x) &= 2\pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \\ &= 2\pi x \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_1H_n}\left(1-\frac{x}{n}\right)^{{}_1H_{-n}}\right). \end{aligned}$$

For example

$$\begin{aligned} S_2(x) &= 2\pi x e^{-x} \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{n+1}\left(1-\frac{x}{n}\right)^{-n+1} e^{-2x}\right) \\ &= 2\pi x e^{-x} \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_2H_n}\left(1-\frac{x}{n}\right)^{{}_2H_{-n}} e^{-2x}\right) \end{aligned}$$

and

$$\begin{aligned} S_3(x) &= 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4}-\frac{3}{2}x} \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n^2}{2}+\frac{3n}{2}+1}\left(1-\frac{x}{n}\right)^{\frac{n^2}{2}-\frac{3n}{2}+1} e^{\frac{x^2}{2}-3x}\right) \\ &= 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4}-\frac{3}{2}x} \prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{{}_3H_n}\left(1-\frac{x}{n}\right)^{{}_3H_{-n}} e^{\frac{x^2}{2}-3x}\right) \end{aligned}$$

(see §2).

Our main results are as follows. The first result expresses the values of the Riemann zeta function at positive odd integers.

Theorem 1.1 *Let $n = 1, 2, 3, \ldots,$ and for $k = 1, 2, \ldots, n$ put*

$$a(2n+1, k) = \sum_{l=1}^{k}(-1)^{k-l} l^{2n}\binom{2n+1}{k-l},$$

which is a positive integer. Then we have:

(1)

$$\zeta'(-2n) = -\log\left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}\right).$$

(2)

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!}\log\left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}\right).$$

Examples 1.2 We have

$$\zeta(3) = 4\pi^2\log S_3(1), \tag{1.1}$$

$$\zeta(5) = -\frac{4\pi^4}{3}\log(S_5(1)S_5(2)^{11}), \tag{1.2}$$

$$\zeta(7) = \frac{8\pi^6}{45}\log(S_7(1)S_7(2)^{57}S_7(3)^{302}). \tag{1.3}$$

The above formula (1.1) was proved in [KK] previously.

Remark 1.3 By the formula

$$S_r(k) = \prod_{l=0}^{k-1} S_{r-l}(1)^{\binom{k-1}{l}(-1)^l} \tag{1.4}$$

for $1 \le k < r$ (cf. §2), we can also express $\zeta(2n+1)$ in terms of $S_l(1)$ $(2 \le l \le 2n+1)$:

$$\zeta(2n+1) = \frac{(-1)^{n-1}2^{2n+1}\pi^{2n}}{(2n)!}\log\left(\prod_{l=2}^{2n+1} S_l(1)^{b(2n+1,l)}\right)$$

with $b(2n+1,l) \in \mathbb{Z}$.

Example 1.4 Since $S_5(2) = S_5(1)S_4(1)^{-1}$ (see §2),

$$\zeta(5) = -\frac{4\pi^4}{3}\log(S_5(1)^{12}S_4(1)^{-11}).$$

Next, let χ be a non-trivial primitive Dirichlet character modulo N, and

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} \tag{1.5}$$

the Dirichlet L-function. Then the values $L(r,\chi)$ for $r = 1, 2, 3, \ldots$ are classified as

$$L(r,\chi) = \begin{cases} \pi^r \cdot (\chi\text{-Bernoulli number}) & \cdots \quad \chi(-1) = (-1)^r \\ \text{“difficult”} & \cdots \quad \chi(-1) = (-1)^{r+1}. \end{cases}$$

Here "difficult" means that these values have not been calculated explicitly yet except for the $r = 1$ case appearing in the Dirichlet's class number formula.

We generalize Dirichlet's result to some difficult case.

Theorem 1.5 *Let χ be a primitive odd character modulo N. Then:*

(1)

$$L'(-1,\chi) = -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\chi(k)}.$$

(2)

$$L(2,\chi) = \frac{2\pi i\tau(\chi)}{N^2}\log\prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\bar{\chi}(k)}.$$

Examples 1.6 We have

$$\begin{aligned}
L(2,\left(\frac{-4}{*}\right)) &= -\frac{\pi}{4}\log\left(S_2\left(\frac{1}{4}\right)^4 S_1\left(\frac{1}{4}\right) S_2\left(\frac{3}{4}\right)^{-4} S_1\left(\frac{3}{4}\right)^{-3}\right) \\
&= \frac{\pi}{4}\log\left(2^3 S_2\left(\frac{1}{4}\right)^{-8}\right), \\
L(2,\left(\frac{-3}{*}\right)) &= -\frac{2\sqrt{3}\pi}{9}\log\left(S_2\left(\frac{1}{3}\right)^3 S_1\left(\frac{1}{3}\right) S_2\left(\frac{2}{3}\right)^{-3} S_1\left(\frac{2}{3}\right)^{-2}\right) \\
&= \frac{4\sqrt{3}\pi}{9}\log\left(3S_2\left(\frac{1}{3}\right)^{-3}\right),
\end{aligned}$$

where we used the following relations (see §2):

$$\begin{aligned}
S_2(1-x) &= S_2(1+x)^{-1} \\
&= \left(S_2(x)S_1(x)^{-1}\right)^{-1} \\
&= S_2(x)^{-1}S_1(x).
\end{aligned}$$

Theorem 1.7 *Let χ be a non-trivial primitive even character modulo N. Then:*

(1)

$$L'(-2,\chi) = -\frac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)}.$$

(2)

$$L(3,\chi)=\frac{2\pi^2\tau(\chi)}{N^3}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2}S_2\left(\frac{k}{N}\right)^{2Nk-3N^2}S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\bar{\chi}(k)}.$$

Example 1.8

$$\begin{aligned}L(3,\left(\frac{12}{*}\right)) &= \frac{\sqrt{3}\pi^2}{432}\log\Bigg(S_3\left(\frac{1}{12}\right)^{288}S_2\left(\frac{1}{12}\right)^{-408}S_1\left(\frac{1}{12}\right)S_3\left(\frac{5}{12}\right)^{-288}S_2\left(\frac{5}{12}\right)^{312}S_1\left(\frac{5}{12}\right)^{-25}\\ &\quad S_3\left(\frac{7}{12}\right)^{-288}S_2\left(\frac{7}{12}\right)^{264}S_1\left(\frac{7}{12}\right)^{-49}S_3\left(\frac{11}{12}\right)^{288}S_2\left(\frac{11}{12}\right)^{-164}S_1\left(\frac{11}{12}\right)^{121}\Bigg).\end{aligned}$$

Thus the values $S_r(a)$ for $a\in\mathbb{Q}$ satisfying $0<a<r$ are quite interesting in relation to zeta values. We formulate our expectation as

Expectation 1.9 $S_r(a)\in\overline{\mathbb{Q}}$ *for* $a\in\mathbb{Q}$ *satisfying* $0<a<r$.

The situation would become transparent when we generalize it as below:

Expectation 1.10 $S_r(\frac{k_1\omega_1+\cdots+k_r\omega_r}{N};\underline{\omega})\in\overline{\mathbb{Q}}$ *for* $N=1,2,3,...$ *and* $k_i=0,1,...,N-1$.

It is easy to see that Expectation 1.9 is contained in Expectation 1.10 for $\underline{\omega}=(1,...,1)$, and Expectation 1.10 clearly indicates that we are studying division values of multiple sine functions.

We note that Shintani [Sh] deeply studied $S_2(x,(1,\varepsilon))$ for a fundamental unit ε of a real quadratic field. In particular, he showed its appearance in the expression for a special value of a suitable L-function, and he obtained certain algebraicity such as

$$S_2\left(\frac{1}{3},(1,\varepsilon)\right)S_2\left(1+\frac{\varepsilon}{3},(1,\varepsilon)\right)S_2\left(\frac{2+2\varepsilon}{3},(1,\varepsilon)\right)=\sqrt{\frac{\frac{1+\sqrt{21}}{2}-\sqrt{\frac{3+\sqrt{21}}{2}}}{2}}$$

for $\varepsilon=\frac{5+\sqrt{21}}{2}$, which is the fundamental unit of $\mathbb{Q}(\sqrt{21})$. Moreover, Shintani studied Kronecker's Jugendtraum for a real quadratic field by using $S_2(x,(1,\varepsilon))$ (he denoted it by $F(x;(1,\varepsilon))^{-1}$). It might be valuable to report the following general product formula

$$\prod_{\substack{k_1,...,k_r=0\\(k_1,...,k_r)\neq(0,...,0)}}^{N-1}S_r\left(\frac{k_1\omega_1+\cdots+k_r\omega_r}{N};(\omega_1,...,\omega_r)\right)=N$$

for an integer $N\geq 2$. (See §2.)

Theorem 1.11 *(1) Expectations 1.9 and 1.10 are valid for* $r=1$.

(2) Expectations 1.9 and 1.10 are valid for $r = 2$ with $N = 2$. Actually

$$S_2\left(\frac{\omega_1}{2};\underline{\omega}\right) = S_2\left(\frac{\omega_2}{2};\underline{\omega}\right) = \sqrt{2}$$

and

$$S_2\left(\frac{\omega_1+\omega_2}{2};\underline{\omega}\right) = 1.$$

Remark 1.12 This paper was referred to in [KK] as a preprint in 2001.

2 Multiple Sine functions

The basic properties of multiple sine functions were proved in [K] and [KK]. Here we recall some of them.

Theorem 2.1 *[KK, Theorem 2.1] The multiple sine function $S_r(x,\underline{\omega})$ satisfies the following identities:*

(a) For $\underline{\omega} = (\omega_1, ..., \omega_r) \in \mathbb{R}_+^r$ put $\underline{\omega}(i) = (\omega_1, ..., \omega_{i-1}, \omega_{i+1}, ..., \omega_r) \in \mathbb{R}_+^{r-1}$, then we have

$$S_r(x+\omega_i,\underline{\omega}) = S_r(x,\underline{\omega})S_{r-1}(x,\underline{\omega}(i))^{-1}, \tag{2.1}$$

where we put $S_0(x,\cdot) \equiv -1$.

(b) For a positive integer N, we have

$$S_r(Nx,\underline{\omega}) = \prod_{0\le k_i\le N-1} S_r\left(x+\frac{\mathbf{k}\cdot\underline{\omega}}{N},\underline{\omega}\right), \tag{2.2}$$

where the product is taken over the vectors $\mathbf{k} = (k_1, ..., k_r)$.

(c)

$$\prod_{\substack{0\le k_i\le N-1\\ \mathbf{k}\neq\mathbf{0}}} S_r\left(\frac{\mathbf{k}\cdot\underline{\omega}}{N},\underline{\omega}\right) = N.$$

(d)

$$S_r(0,\underline{\omega}) = 0.$$

(e) We have for any $c > 0$ the homogeneity

$$S_r(cx,c\underline{\omega}) = S_r(x,\underline{\omega}).$$

Theorem 2.2 *(a) For $r \geq 2$ we have*

$$S_r(x+1) = S_r(x) S_{r-1}(x)^{-1}.$$

(b)

$$S_r(2x) = \prod_{k=0}^{r} S_r\left(x + \frac{k}{2}\right)^{\binom{r}{k}}.$$

(c) Put $Q_r(x) = (-1)^{r-1}\pi\binom{x-1}{r-1}$, then

$$\frac{S_r'(x)}{S_r(x)} = Q_r(x)\cot(\pi x).$$

(d)

$$S_r''(x) + \left(\pi Q_r(x)^{-1} - 1\right) S_r'(x)^2 S_r(x)^{-1} - Q_r'(x) Q_r(x)^{-1} S_r'(x) + \pi Q_r(x) S_r(x) = 0.$$

(e)

$$\begin{aligned} S_2(x) &= 2\pi x e^{-x} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{n}\right)^{n+1} \left(1-\frac{x}{n}\right)^{-n+1} e^{-2x} \right) \\ &= 2\pi x e^{-x} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{n}\right)^{\,_2H_n} \left(1-\frac{x}{n}\right)^{\,_2H_{-n}} e^{-2x} \right). \end{aligned}$$

(f)

$$\begin{aligned} S_3(x) &= 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4} - \frac{3}{2}x} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{n}\right)^{\frac{n^2}{2}+\frac{3n}{2}+1} \left(1-\frac{x}{n}\right)^{\frac{n^2}{2}-\frac{3n}{2}+1} e^{\frac{x^2}{2}-3x} \right) \\ &= 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4} - \frac{3}{2}x} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{n}\right)^{\,_3H_n} \left(1-\frac{x}{n}\right)^{\,_3H_{-n}} e^{\frac{x^2}{2}-3x} \right). \end{aligned}$$

Proof. The assertions (a) and (b) are immediate consequences from [KK, Theorem 2.1]. The differential equation (c) is proved in [KK, Theorem 2.15]. We compute from (c) that

$$\begin{aligned} \left(Q_r(x)^{-1}\frac{S_r'}{S_r}(x)\right)' &= (\cot \pi x)' \\ &= -\frac{\pi}{\sin^2 \pi x} \\ &= -\pi(\cot^2(\pi x) + 1) \\ &= -\pi\left(\left(Q_r(x)^{-1}\frac{S_r'}{S_r}(x)\right)^2 + 1\right), \end{aligned}$$

which gives the proof of (d). Finally (e) and (f) are deduced from [KK, Examples 3.6], where we express the normalized multiple sine functions $S_r(x)$ in terms of primitive multiple sine functions which are defined by the Hadamard product. ∎

3 The Riemann zeta function

Lemma 3.1 *There exist uniquely determined integers* $a(r,k)$ *such that*

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{x-k} \tag{3.1}$$

with ${}_rH_{x-k} = \frac{(x-k+r-1)\cdots(x-k+1)}{(r-1)!}$ *for an indeterminate* x. *Indeed* $a(r,k)$ *are given as follows:*

$$a(r,k) = \sum_{l=1}^{k} (-1)^{k-l} l^{r-1} \binom{r}{k-l}. \tag{3.2}$$

Moreover,

$$a(r, r-k) = a(r,k). \tag{3.3}$$

Proof. The existence of $a(r,k)$ follows from the fact that the $(r-1)$ polynomials ${}_rH_{x-k}$ $(k = 1, ..., r-1)$ are linearly independent over $\mathbb{Q}$. By putting $x = k$ in (3.1), we have

$$k^{r-1} = a(r,1)\binom{k+r-2}{r-1} + a(r,2)\binom{k+r-3}{r-1} + \cdots + a(r,k)\cdot 1.$$

This leads to

$$a(r,k) = k^{r-1} - \sum_{j=1}^{k-1} a(r,j)\binom{k+r-1-j}{r-1}.$$

Thus (3.2) is proved by induction on k. Next, from (3.1)

$$(-x)^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{-x-k}$$

and

$${}_rH_{-x-k} = \frac{(-x-k+r-1)\cdots(-x-k+1)}{(r-1)!} = (-1)^{r-1}\ {}_rH_{x-(r-k)},$$

so

$$x^{r-1} = \sum_{k=1}^{r-1} a(r,k)\ {}_rH_{x-(r-k)} = \sum_{k=1}^{r-1} a(r,r-k)\ {}_rH_{x-k}.$$

Hence, by the uniquenes of $a(r,k)$ we have $a(r,r-k) = a(r,k)$. ∎

Examples 3.2 For $x = n \in \mathbb{Z}$ and $r = 2, 3, 4, 5$ we have

$$\begin{aligned} n &= {}_2H_{n-1}, \\ n^2 &= {}_3H_{n-1} + {}_3H_{n-2}, \\ n^3 &= {}_4H_{n-1} + 4\ {}_4H_{n-2} + {}_4H_{n-3}, \\ n^4 &= {}_5H_{n-1} + 11\ {}_5H_{n-2} + 11\ {}_5H_{n-3} + {}_5H_{n-4}. \end{aligned}$$

Proof of Theorem 1.1:
For $r \geq 2$ we have by Lemma 3.1

$$\begin{aligned} \zeta(s+1-r) &= \sum_{n=1}^{\infty} \frac{n^{r-1}}{n^s} \\ &= \sum_{k=1}^{r-1} a(r,k) \sum_{n=1}^{\infty} \frac{{}_rH_{n-k}}{n^s} \\ &= \sum_{k=1}^{r-1} a(r,k) \zeta_r(s,k), \end{aligned}$$

where $\zeta_r(s,k)$ is the multiple Hurwitz zeta function

$$\zeta_r(s,k) = \sum_{n=0}^{\infty} \frac{{}_rH_n}{(n+k)^s}.$$

Thus we have

$$\zeta'(1-r) = \sum_{k=1}^{r-1} a(r,k) \log \Gamma_r(k).$$

In case $r = 2n+1$, it follows that

$$\begin{aligned} \zeta'(-2n) &= \sum_{k=1}^{2n} a(2n+1,k) \log \Gamma_{2n+1}(k) \\ &= -\sum_{k=1}^{n} a(2n+1,k) \log S_{2n+1}(k) \\ &= -\log \left(\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)} \right), \end{aligned}$$

where we used $S_{2n+1}(k) = \Gamma_{2n+1}(k)^{-1}\Gamma_{2n+1}(2n+1-k)^{-1}$ and $a(2n+1, 2n+1-k) = a(2n+1,k)$.∎

Examples 3.3 We saw in [KK, Theorem 3.8(c)] that

$$\zeta(3) = 4\pi^2 \log S_3(1).$$

Combining this with the fact that

$$S_3(1) = \sqrt{2} S_3\left(\frac{1}{2}\right)^{-4/3},$$

which can be obtained by the facts

$$\begin{aligned} S_3(1) &= S_3(2 \cdot \frac{1}{2}) \\ &= S_3(\frac{1}{2}) S_3(1)^3 S_3(\frac{3}{2})^3 S_3(2) \\ &= S_3(1)^4 S_3(\frac{1}{2})^4 S_2(\frac{1}{2})^{-3} \end{aligned}$$

and that $S_2(\frac{1}{2}) = \sqrt{2}$, we have

$$\zeta(3) = \frac{16\pi^2}{3} \log \left(S_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{3}{8}} \right)$$

which was proved in [KK, Theorem 3.8(b)] by another method (using a "primitive multiple sine function").

4 Dirichlet L-functions for odd characters

We prove the formula for $L(2,\chi)$ for odd characters. Since our method follows a proof for Dirichlet's result on $L(1,\chi)$ for even characters, we first recall it. We show the formula for $L'(0,\chi)$. Then the result on $L(1,\chi)$ follows via the functional equation.

Let χ be a non-trivial primitive Dirichlet character modulo N. We have

$$\begin{aligned} L(s,\chi) &= \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \frac{1}{(mN+k)^s} \\ &= N^{-s} \sum_{k=1}^{N-1} \chi(k) \zeta(s, \frac{k}{N}), \end{aligned}$$

where

$$\zeta(s,x) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^s}$$

is the Hurwitz zeta function. Hence

$$L(0,\chi) = \sum_{k=1}^{N-1} \chi(k)\zeta(0, \frac{k}{N})$$

and

$$\begin{aligned}
L'(0,\chi) &= \sum_{k=1}^{N-1} \chi(k)\zeta'(0, \frac{k}{N}) - (\log N) \sum_{k=1}^{N-1} \chi(k)\zeta(0, \frac{k}{N}) \\
&= \sum_{k=1}^{N-1} \chi(k)\zeta'(0, \frac{k}{N}) - (\log N) L(0,\chi).
\end{aligned}$$

When χ is even, it holds that $L(0,\chi) = 0$ (this is the reason of "difficult"), so we have

$$\begin{aligned}
L'(0,\chi) &= \sum_{k=1}^{N-1} \chi(k)\zeta'(0, \frac{k}{N}) \\
&= \sum_{k=1}^{N-1} \chi(k) \log \Gamma_1 \left(\frac{k}{N}\right) \\
&= \frac{1}{2} \sum_{k=1}^{N-1} \chi(k) \left(\log \Gamma_1 \left(\frac{k}{N}\right) + \log \Gamma_1 \left(\frac{N-k}{N}\right)\right) \\
&= -\frac{1}{2} \sum_{k=1}^{N-1} \chi(k) \log S_1 \left(\frac{k}{N}\right).
\end{aligned}$$

This gives the Dirichlet's result.

Proof of Theorem 1.5

We prove (1), then (2) is obtained via the functional equation. Since

$$\zeta(s-1,x) = \sum_{n=0}^{\infty} \frac{n+x}{(n+x)^s} = \sum_{n=0}^{\infty} \frac{n+1}{(n+x)^s} + (x-1) \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \zeta_2(s,x) + (x-1)\zeta_1(s,x),$$

we have

$$\zeta'(-1,x) = \zeta_2'(0,x) + (x-1)\zeta_1'(0,x),$$

as $\zeta_1(s,x) = \zeta(s,x)$. Now that χ is odd and that $L(-1,\chi) = 0$, we compute

$$\begin{aligned}
L'(-1,\chi) &= N\sum_{k=1}^{N-1}\chi(k)\zeta'(-1,\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\zeta_2'(0,\tfrac{k}{N}) + N\sum_{k=1}^{N-1}\chi(k)(\tfrac{k}{N}-1)\zeta_1'(0,\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\log\Gamma_2(\tfrac{k}{N}) + N\sum_{k=1}^{N-1}\chi(k)(\tfrac{k}{N}-1)\log\Gamma_1'(\tfrac{k}{N}) \\
&= N\sum_{k=1}^{N-1}\chi(k)\log\left(\Gamma_2(\tfrac{k}{N})\Gamma_1(\tfrac{k}{N})^{\frac{k}{N}-1}\right) \\
&= \frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(\frac{\Gamma_2(\frac{k}{N})}{\Gamma_2(1-\frac{k}{N})}\frac{\Gamma_1(\frac{k}{N})^{\frac{k}{N}-1}}{\Gamma_1(1-\frac{k}{N})^{-\frac{k}{N}}}\right) \\
&= \frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(\frac{\Gamma_2(\frac{k}{N})}{\Gamma_2(2-\frac{k}{N})}\left(\Gamma_1(\tfrac{k}{N})\Gamma_1(1-\tfrac{k}{N})\right)^{\frac{k}{N}-1}\right) \\
&= -\frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(S_2(\tfrac{k}{N})S_1(\tfrac{k}{N})^{\frac{k}{N}-1}\right) \\
&= -\frac{N}{2}\sum_{k=1}^{N-1}\chi(k)\log\left(S_2(\tfrac{k}{N})S_1(\tfrac{k}{N})^{\frac{k}{N}}\right),
\end{aligned}$$

where we used the fact $S_1(\frac{k}{N}) = S_1(\frac{N-k}{N})$ with $\chi(N-k) = -\chi(k)$. ∎

5 Dirichlet L-functions for even characters

Proof of Theorem 1.7

We again show (1), then (2) is obtained via the functional equation. Since

$$(n+x)^2 = 2\ {}_3H_n + (2x-3)\ {}_2H_n + (x-1)^2\ {}_1H_n,$$

we have

$$\zeta(s-2,x) = \sum_{n=0}^{\infty}\frac{(n+x)^2}{(n+x)^s} = 2\zeta_3(s,x) + (2x-3)\zeta_2(s,x) + (x-1)^2\zeta_1(s,x).$$

Therefore we have

$$\zeta'(-2,x) = 2\zeta_3'(0,x) + (2x-3)\zeta_2'(0,x) + (x-1)^2\zeta_1'(0,x).$$

Now that χ is even and that $L(-2,\chi)=0$, we compute

$$\begin{aligned}
L'(-2,\chi) &= N^2\sum_{k=1}^{N-1}\chi(k)\zeta'(-2,\tfrac{k}{N}) \\
&= N^2\sum_{k=1}^{N-1}\chi(k)\left(2\zeta_3'(0,\tfrac{k}{N})+(2\tfrac{k}{N}-3)\zeta_2'(0,\tfrac{k}{N})+(\tfrac{k}{N}-1)^2\zeta_1'(0,\tfrac{k}{N})\right) \\
&= N^2\sum_{k=1}^{N-1}\chi(k)\log\left(\Gamma_3(\tfrac{k}{N})^2\Gamma_2(\tfrac{k}{N})^{2\frac{k}{N}-3}\Gamma_1(\tfrac{k}{N})^{(\frac{k}{N}-1)^2}\right) \\
&= -\tfrac{1}{2}\log\prod_{k=1}^{N-1}\left(S_3(\tfrac{k}{N})^{2N^2}S_2(\tfrac{k}{N})^{2Nk-3N^2}S_1(\tfrac{k}{N})^{k^2}\right)^{\chi(k)}. \blacksquare
\end{aligned}$$

6 Division values of normalized multiple sines

Proof of Theorem 1.11

Since $S_1(x,\omega)=2\sin(\frac{\pi x}{\omega})$ by [KK, §2], we have

$$S_1(\tfrac{k\omega}{N},\omega)=2\sin(\tfrac{k\pi}{N})=-i(e^{i\pi k/N}-e^{-i\pi k/N})\in\overline{\mathbb{Q}},$$

which leads to (1).

Recall that

$$S_2(x,(\omega_1,\omega_2))=\frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}.$$

First

$$S_2(\tfrac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))=\frac{\Gamma_2(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))}{\Gamma_2(\frac{\omega_1+\omega_2}{2},(\omega_1,\omega_2))}=1.$$

Secondly

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))=\frac{\Gamma_2(\frac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2))}{\Gamma_2(\frac{\omega_1}{2},(\omega_1,\omega_2))}.$$

Here we use ([KK, §2])

$$\Gamma_2(x+\omega_2,(\omega_1,\omega_2))=\Gamma_2(x,(\omega_1,\omega_2))\Gamma_1(x,\omega_1)^{-1}.$$

Then

$$\Gamma_2(\tfrac{\omega_1}{2}+\omega_2,(\omega_1,\omega_2))=\Gamma_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))\Gamma_1(\tfrac{\omega_1}{2},\omega_1)^{-1}.$$

Hence

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2))=\Gamma_1(\tfrac{\omega_1}{2},\omega_1)^{-1}.$$

Now ([KK, §2])

$$\Gamma_1(x,\omega) = \frac{\Gamma(\frac{x}{\omega})}{\sqrt{2\pi}}\omega^{\frac{x}{\omega}-\frac{1}{2}},$$

so

$$\Gamma_1(\tfrac{\omega_1}{2},\omega_1) = \frac{\Gamma(\frac{1}{2})}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}.$$

Thus

$$S_2(\tfrac{\omega_1}{2},(\omega_1,\omega_2)) = \sqrt{2}. \blacksquare$$

Remark 6.1 A suitable restriction on the form of division points such as made in Expectation 1.10 will be needed as the following example shows:

$$S_2(2,(1,\sqrt{2})) \notin \overline{\mathbb{Q}}. \tag{6.1}$$

By this example, we must seriously look at $S_r(a_1\omega_1 + \cdots + a_r\omega_r;(\omega_1,\cdots,\omega_r))$ for general $a_i \in \mathbb{Q}$. The proof of (6.1) is given by

$$\frac{S_2(2,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))} = \frac{S_2(1+1,(1,\sqrt{2}))}{S_2(1,(1,\sqrt{2}))} = S_1(1,\sqrt{2})^{-1} = \left(2\sin\frac{\pi}{\sqrt{2}}\right)^{-1} \notin \overline{\mathbb{Q}},$$

where

$$\begin{aligned} 2\sin\frac{\pi}{\sqrt{2}} &= -i(e^{i\frac{\pi}{\sqrt{2}}} - e^{-i\frac{\pi}{\sqrt{2}}}) \\ &= -i((-1)^{1/\sqrt{2}} - ((-1)^{1/\sqrt{2}})^{-1}) \end{aligned}$$

and we used the transcendency result of Gelfond-Schneider $(-1)^{1/\sqrt{2}} \notin \overline{\mathbb{Q}}$. Moreover we appeal to the following facts:

$$\begin{aligned} S_2(\omega_1,(\omega_1,\omega_2)) &= \sqrt{\frac{\omega_2}{\omega_1}}, \\ S_2(\omega_2,(\omega_1,\omega_2)) &= \sqrt{\frac{\omega_1}{\omega_2}}. \end{aligned}$$

In particular

$$S_2(1,(1,\sqrt{2})) = 2^{\frac{1}{4}} \in \overline{\mathbb{Q}}.$$

Thus we obtain (6.1).

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