Multiple Euler Products

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Dedicated to Professor Sergei Vostokov

Running title. Multiple Euler Products

Abstract. We construct a new zeta function having zeros or poles at sums of zeros or poles of Euler products. Multiple Euler factors and multiple Euler products are explicitly calculated in basic cases.

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1 Introduction

In 1737, Leonhard Euler [E] made a quite important discovery at St Petersburg on the existence of the Euler product expression for the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where p runs over the prime numbers. A typical consequence of this Euler product is the explicit formula for the number $\pi(x)$ of primes under x given by Riemann [R] using the zeros and the pole of $\zeta(s)$. Up to now it is known that there exist various zeta functions having Euler product expressions.

The purpose of this paper is to study multiple Euler factors (MEF) and multiple Euler products (MEP) via multiple explicit formulas (MEF) naturally associated to "absolute tensor products" of several zeta functions.

We recall the construction of the absolute tensor product (ATP). We refer to [KK1] and [KW1] for details.

Let

$$\begin{aligned} Z_j(s) &= \prod_{\rho\in\mathbf{C}} (s-\rho)^{m_j(\rho)} \\ &= \exp\left(-\frac{\partial}{\partial w}\bigg|_{w=0} \sum_{\rho\in\mathbf{C}} \frac{m_j(\rho)}{(s-\rho)^w} \right) \end{aligned}$$

be "zeta functions" expressed as regularized product, where

$$m_j \ : \ \mathbf{C} \to \mathbf{Z}$$

denotes the multiplicity function for $j = 1, ..., r$. The absolute tensor product $(Z_1 \otimes \cdots \otimes Z_r)(s)$ is defined as

$$(Z_1 \otimes \cdots \otimes Z_r)(s) = \prod_{\rho_1,\cdots,\rho_r\in\mathbf{C}} (s-(\rho_1+\cdots+\rho_r))^{m(\rho_1,\cdots,\rho_r)}$$

with

$$m(\rho_1,\cdots,\rho_r) = m_1(\rho_1)\cdots m_r(\rho_r) \times \begin{cases} 1 & \mathrm{Im}(\rho_j) \geq 0, \quad (j=1,...,r) \\ (-1)^{r-1} & \mathrm{Im}(\rho_j) < 0, \quad (j=1,...,r) \\ 0 & \text{otherwise.} \end{cases}$$

This definition originates from [K]. We refer to the excellent survey of Manin [M]. The notation of the regularized product is due to Deninger [Den1]. See [HKW] concerning the needed regularized products. The absolute tensor product was studied by Schröter [S] in the name of the "Kurokawa tensor product."

We are especially interested in the case of Hasse zeta functions $Z_j(s) = \zeta(s, A_j)$ for commutative rings $A_1, ..., A_r$ of finite type over $\mathbf{Z}$. We recall that the Hasse zeta function $\zeta(s, A)$ of a commutative ring A is defined to be

$$\begin{aligned} \zeta(s,A) &= \prod_{\mathbf{m}} (1-N(\mathbf{m})^{-s})^{-1} \\ &= \exp\left(\sum_{\mathbf{m}} \sum_{k=1}^{\infty} \frac{1}{k} N(\mathbf{m})^{-ks} \right), \end{aligned}$$

where $\mathbf{m}$ runs over maximal ideals of A and $N(\mathbf{m}) = \#(A/\mathbf{m})$. This is also written as

$$\zeta(s,A) = \exp\left(\sum_{p:\text{primes}} \sum_{m=1}^{\infty} \frac{\#\mathrm{Hom}_{\mathrm{ring}}(A, \mathbf{F}_{p^m})}{m} p^{-ms} \right).$$

For simplicity we write

$$\zeta(s, A_1 \otimes \cdots \otimes A_r) = \zeta(s,A_1) \otimes \cdots \otimes \zeta(s,A_r).$$

Actually, as was explained by Manin [M], we expect that our multiple zeta function would be the zeta function of the "absolute tensor product"

$$A_1 \otimes_{\mathbf{F}_1} \cdots \otimes_{\mathbf{F}_1} A_r$$

that is the tensor product over the (virtual) "one element field" $\mathbf{F}_1$. See [KOW] and [Dei] for the absolute mathematics over "$\mathbf{F}_1$". In any way, we notice that $\zeta(s, A_1 \otimes \cdots \otimes A_r)$ has the following additive structure on zeros and poles: if $\zeta(s_j, A_j) = 0$ or ∞ and $\mathrm{Im}(s_j)$ $(j = 1, ..., r)$ have the same signature, then $\zeta(s_1 + \cdots + s_r, A_1 \otimes \cdots \otimes A_r) = 0$ or ∞.

Such an additive structure was crucial in the study of Hasse zeta functions of positive characteristic (congruence zeta functions) pursued by Grothendieck [G] and Deligne [D], where Euler products were important to restrict the region of zeros and poles for our reaching to the analogue of the Riemann Hypothesis.

We expect that our multiple zeta functions also have Euler products of the following form:

$$\zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r) = \prod_{(\mathbf{m}_1, ..., \mathbf{m}_r)} H_{(\mathbf{m}_1,...,\mathbf{m}_r)}(N(\mathbf{m}_1)^{-s}, ..., N(\mathbf{m}_1)^{-s})$$

where $\mathbf{m}_i$ runs over the maximal ideals of A_i and $H_{(\mathbf{m}_1,...,\mathbf{m}_r)}(T_1, ..., T_r)$ is a power series in T_1,...,T_r of the constant term 1 with a possible degeneration at $(\mathbf{m}_1, ..., \mathbf{m}_r)$, where $N(\mathbf{m}_i) = N(\mathbf{m}_j)$ for some $i \neq j$. More generally we expect that the multiple zeta function $Z_1(s) \otimes \cdots \otimes Z_r(s)$ has an Euler product

$$Z_1(s) \otimes \cdots \otimes Z_r(s) = \prod_{(p_1,...,p_r) \in P_1 \times \cdots \times P_r} H_{(p_1,...,p_r)}(N(p_1)^{-s}, ..., N(p_r)^{-s})$$

when each zeta function $Z_j(s)$ has an Euler product

$$Z_j(s) = \prod_{p \in P_j} H_p^j(N(p)^{-s})$$

and a functional equation; here $H_p^j(T)$ is a power series in T and $H_{(p_1,...,p_r)}(T_1, ..., T_r)$ is a power series in $(T_1, ..., T_r)$ with a possible degeneration at $(p_1, ..., p_r)$, where $N(p_i) = N(p_j)$ for some $i \neq j$.

In a previous paper [KK1] we investigated the absolute tensor product $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ for primes p and q by using a signed double Poisson summation formula, where $\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}$. In other words we constructed a new zeta function having zeros (or poles) at sums of poles of $\zeta(s, \mathbf{F}_p)$ and $\zeta(s, \mathbf{F}_q)$. We state the result as follows in refining and complementing the main theorem of [KK1]. For simplicity we use the notation $F(s) \cong G(s)$ for functions $F(s)$ and $G(s)$ to indicate that $F(s) = e^{Q(s)}G(s)$ for some polynomial $Q(s)$.

Theorem 1 *Let p and q be distinct prime numbers. Define the function $\zeta_{p,q}(s)$ in $\mathrm{Re}(s) > 0$ as follows:*

$$\begin{aligned}\zeta_{p,q}(s) \quad := \quad \exp\Bigg(& -\frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log p}{\log q}\right)}{n}p^{-ns} - \frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log q}{\log p}\right)}{n}q^{-ns} \\ & -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}q^{-ns}\Bigg).\end{aligned}$$

Then the function $\zeta_{p,q}(s)$ has the following properties:

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,q}(s)$ has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function of order two.

(2) All zeros and poles of $\zeta_{p,q}(s)$ are simple and located at

$$s = 2\pi i\left(\frac{m}{\log p} + \frac{n}{\log q}\right),$$

where (m, n) is a pair of nonnegative integers or a pair of negative integers. Indeed it gives a zero or pole according as they are nonnegative or negative.

(3) We have the identification

$$\zeta_{p,q}(s) \cong \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q).$$

(4) The function $\zeta_{p,q}(s)$ satisfies a functional equation:

$$\begin{aligned}\zeta_{p,q}(-s) = \zeta_{p,q}(s)^{-1}(pq)^{\frac{s}{2}}(1-p^{-s})(1-q^{-s}) & \\ \times \exp\left(\frac{i\log p\log q}{4\pi}s^2 - \frac{\pi i}{6}\left(\frac{\log q}{\log p} + \frac{\log p}{\log q} + 3\right)\right).\end{aligned}$$

When $p = q$ the result is as follows:

Theorem 2 *Let*

$$\zeta_{p,p}(s) := \exp\left(\frac{i}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns} - \left(1 - \frac{i\log p}{2\pi}s\right)\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right)$$

in $\mathrm{Re}(s) > 0$. Then the function $\zeta_{p,p}(s)$ has the following properties:

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,p}(s)$ *has an analytic continuation to all* $s \in \mathbf{C}$ *as a meromorphic function of order two.*

(2) All zeros and poles of $\zeta_{p,p}(s)$ *are located at*

$$s = \frac{2\pi i n}{\log p},$$

which gives a zero or pole of order $|n+1|$, *according as* n *is a nonnegative or negative integer.*

(3) We have the identification

$$\zeta_{p,p}(s) \cong \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p).$$

(4) The function $\zeta_{p,p}(s)$ *satisfies a functional equation:*

$$\zeta_{p,p}(-s) = \zeta_{p,p}(s)^{-1} p^s (1-p^{-s})^2 \exp\left(\frac{i(\log p)^2}{4\pi}s^2 - \frac{5\pi i}{6}\right).$$

Remark 1.1 These theorems are natural generalizations of the following well-known facts on the simplest zeta function

$$\zeta_p(s) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} p^{-ms}\right)$$

defined in $\mathrm{Re}(s) > 0$.

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_p(s)$ has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function of order one. In fact, $\zeta_p(s) = (1-p^{-s})^{-1}$ for all $s \in \mathbf{C}$.

(2) The function $\zeta_p(s)$ has no zeros. All poles of $\zeta_p(s)$ are simple and located at $s = 2\pi i n / \log p$ for $n \in \mathbf{Z}$.

(3) We have the identification
$$\zeta_p(s) = \zeta(s, \mathbf{F}_p).$$

(4) The function $\zeta_p(s)$ satisfies a functional equation $\zeta_p(-s) = \zeta_p(s)(-p^{-s})$.

Remark 1.2 The functional equation in Theorem 2(4) "coincides" with Theorem 1(4) when $p = q$. This is remarkable considering the quite different appearance of $\zeta_{p,q}(s)$ and $\zeta_{p,p}(s)$.

Remark 1.3 Let

$$\mathrm{Li}_r(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^r}$$

be the polylogarithm function of order r. Then the function $\zeta_p(s)$ in Remark 1.1 is written as

$$\zeta_p(s) = \exp\left(\mathrm{Li}_1(p^{-s})\right)$$

in $\mathrm{Re}(s) > 0$ since $\mathrm{Li}_1(u) = \log(\frac{1}{1-u})$ for $|u| < 1$. Similarly the function $\zeta_{p,p}(s)$ in Theorem 2 is written as

$$\zeta_{p,p}(s) = \exp\left(\frac{i}{2\pi}\mathrm{Li}_2(p^{-s}) - \left(1 - \frac{i\log p}{2\pi}s\right)\mathrm{Li}_1(p^{-s})\right)$$

in $\mathrm{Re}(s) > 0$. Appearance of such a polylogarithm is characteristic in our multiple Euler factors. The function

$$\sum_{m=1}^{\infty} \frac{\cot(\pi m\alpha)}{m} u^m$$

appearing in Theorem 1 is considered as a variation of a polylogarithm (of order 2); see [KW1] for such a "multiple polylogarithm" (multi-q-log) and its relation to Appell's O-function.

We prove Theorems 1 and 2 in §2 supplementing [KK1], where the signed double Poisson summation formula and the theory of multiple sine functions developed in [KK2] are essential. The main difference from [KK1] is the functional equation not stated there.

From our viewpoint, it is very interesting to see the nature of $\prod_{p,q} \zeta_{p,q}(s)$. Unfortunately, however, it does not converge even for sufficiently large $\mathrm{Re}(s)$. Our "α-version" $\zeta^{\alpha}_{p,q}(s)$ treated below remedies the situation. In passing we notice on the analyticity of the diagonal Euler product shown in §2.

Theorem 3 *Let*

$$Z(s) = \prod_p \zeta_{p,p}(s).$$

Then, $Z(s)$ *is absolutely convergent in* $\mathrm{Re}(s) > 1$, *and it has an analytic continuation with singularities to* $\mathrm{Re}(s) > 0$ *with the natural boundary* $\mathrm{Re}(s) = 0$.

In the later half of this paper we study "the double Riemann zeta function" $\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z})$ by establishing the signed double explicit formula in the following theorem, which generalizes the signed double Poisson summation formula used in the proof of Theorems 1 and 2.

For simplicity put $\xi(s) = \hat{\zeta}(s + \frac{1}{2}) = \hat{\zeta}(s + \frac{1}{2}, \mathbf{Z})$ for $\hat{\zeta}(s) = \Gamma_{\mathbf{R}}(s)\zeta(s)$ with $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. The functional equation of $\zeta(s)$ is written as $\xi(s) = \xi(-s)$. We recall that nontrivial zeros of $\zeta(s)$ are zeros in the strip $|\mathrm{Re}(s - \frac{1}{2})| < 1/2$. We denote by $\frac{1}{2} + i\gamma$ such a zero, where γ is a complex number in $-1/2 < \mathrm{Im}(\gamma) < 1/2$.

Hereafter let $h(t)$ be an odd regular function in $|\mathrm{Re}(t)| < 1$ satisfying $h(t) = O(|t|^{-3})$ as $|t| \to \infty$. We put $H_\alpha(t) := h(2\alpha + it)$ and

$$\widetilde{H}(u) := \int_{-\infty}^{\infty} H(t) e^{itu} dt.$$

Theorem 4 *Let $1/2 < \alpha < 1$. We have*

$$\sum_{\mathrm{Re}(\gamma_1), \mathrm{Re}(\gamma_2) > 0} H_0(\gamma_1 + \gamma_2) = \sum_{\substack{p,q \\ p \neq q}} \mathcal{H}_{p,q}^\alpha + \sum_p \mathcal{H}_{p,p}^\alpha + \sum_p \mathcal{H}_{p,\infty}^\alpha + \mathcal{H}_{\infty,\infty}^\alpha + \mathcal{H}_0^\alpha,$$

where the sum in the left hand side is taken over pairs $(\frac{1}{2} + i\gamma_1,\ \frac{1}{2} + i\gamma_2)$ of nontrivial zeros of the Riemann zeta function, the sum in the right hand side is taken over pairs of distinct primes p, q or primes p, and we define for pairs of distinct primes p, q as

$$\begin{aligned} \mathcal{H}_{p,q}^\alpha &= \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(p^m q^n)} \frac{1}{p^{m(\alpha+\frac{1}{2})} q^{n(\alpha+\frac{1}{2})}} \left(\widetilde{H_0}(-m \log p) + \widetilde{H_0}(-n \log q) \right) && (1.1) \\ &+ \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(\frac{p^m}{q^n})} \frac{1}{p^{m(\alpha+\frac{1}{2})} q^{n(\alpha+\frac{1}{2})}} \left(\widetilde{H_\alpha}(-m \log p) - \widetilde{H_\alpha}(-n \log q) \right), && (1.2) \end{aligned}$$

and for a prime p,

$$\begin{aligned} \mathcal{H}_{p,p}^\alpha &= \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p}{(m+n) p^{(m+n)(\alpha+\frac{1}{2})}} \left(\widetilde{H_0}(-m \log p) + \widetilde{H_0}(-n \log p) \right) && (1.3) \\ &+ \frac{i}{4\pi^2} \sum_{m \neq n} \frac{\log p}{(m-n) p^{(m+n)(\alpha+\frac{1}{2})}} \left(\widetilde{H_\alpha}(-m \log p) - \widetilde{H_\alpha}(-n \log p) \right) && (1.4) \\ &+ \frac{1}{4\pi^2} (\log p)^2 \sum_{m=1}^{\infty} p^{-2m(\alpha+\frac{1}{2})} \widetilde{t H_\alpha(t)}(-m \log p), && (1.5) \end{aligned}$$

$$\begin{aligned} \mathcal{H}_{p,\infty}^\alpha &= -\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\log p}{p^{m(\alpha+\frac{1}{2})}} \int_{-\infty}^{\infty} p^{-imt} H_\alpha(t) \int_0^t p^{imt'} \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} + it' \right) dt' dt && (1.6) \\ &- \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\log p}{p^{m(\alpha+\frac{1}{2})}} \int_{-\infty}^{\infty} H_0(t) \int_0^t \mathrm{Re} \left(p^{im(t-t')} \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} + it' \right) \right) dt' dt, && (1.7) \end{aligned}$$

$$\begin{aligned} \mathcal{H}_{\infty,\infty}^\alpha &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_\alpha(t) \int_0^t \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} + it_1 \right) \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} + i(t - t_1) \right) dt_1 dt && (1.8) \\ &+ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_0(t) \int_0^t \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} + it_1 \right) \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}} \left(\alpha + \frac{1}{2} - i(t - t_1) \right) dt_1 dt, && (1.9) \end{aligned}$$

and

$$\mathcal{H}_0^\alpha = -\frac{\alpha}{\pi}\int_0^\pi \sum_{\mathrm{Re}(\gamma_1)>0} h(i\gamma_1+\alpha e^{i\theta})\frac{\xi'}{\xi}(\alpha e^{i\theta})e^{i\theta}d\theta \tag{1.10}$$

$$-\frac{\alpha^2}{4\pi^2}\int_0^\pi\int_0^\pi h(\alpha e^{i\theta_1}+\alpha e^{i\theta_2})\frac{\xi'}{\xi}(\alpha e^{i\theta_1})\frac{\xi'}{\xi}(\alpha e^{i\theta_2})e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2, \tag{1.11}$$

where $m, n \in \mathbf{Z}$*,* $m, n \geq 1$*.*

Notice that only pairs of zeros in the upper (or lower) half plane are counted in the left hand side of Theorem 4. The method of Cramér [C] is important in the proof: see Deninger [Den2] and Voros [V] around Cramér's method.

For defining the (p,q)-Euler factors, we put

$$h(t) = \frac{1}{(t+s)^2} - \frac{1}{(t-s)^2} \tag{1.12}$$

in Theorem 4. We denote by p, q any (finite or infinite) places. The (p,q)-Euler factor of the double Riemann zeta function $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ is defined as follows:

$$\zeta_{p,q}^\alpha(s+1) = \exp\left(\iint \mathcal{H}_{p,q}^\alpha(s)dsds\right).$$

We also denote the remainder factor by

$$\zeta_0^\alpha(s+1) = \exp\left(\iint \mathcal{H}_0^\alpha(s)dsds\right).$$

Here we notice that these definitions of $\zeta_{p,q}^\alpha(s)$ and $\zeta_0^\alpha(s)$ have ambiguities emerged from the integral constants, that is the factor $\exp(Q(s))$ with $Q(s)$ a polynomial with $\deg(Q)\leq 2$. We do not normalize them at first. Rather we normalize them a posteriori after the calculation; see the remark to Theorem 5 below. The double Riemann zeta function $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ is expressed by an Euler product over the pairs of places (p,q).

Theorem 5 *The* (p,q)*-Euler factors of the double Riemann zeta function* $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ *are described as follows:*

(1) For distinct prime numbers p *and* q*, we have*

$$\begin{aligned}\zeta_{p,q}^\alpha(s) \cong \exp\Bigg(&\frac{1}{\pi i}\sum_{m,n}\frac{(\log p)(\log q)}{(m\log p)^2-(n\log q)^2}\\ &\Bigg(\frac{\cosh(m\alpha\log p)}{q^{n(\alpha+\frac12)}}p^{-m(s-\frac12)}+\frac{n\log q}{m\log p}\frac{\sinh(m\alpha\log p)}{q^{n(\alpha+\frac12)}}p^{-m(s-\frac12)}\\ &-\frac{m\log p}{n\log q}\frac{\sinh(n\alpha\log q)}{p^{m(\alpha+\frac12)}}q^{-n(s-\frac12)}-\frac{\cosh(n\alpha\log q)}{p^{m(\alpha+\frac12)}}q^{-n(s-\frac12)}\Bigg)\Bigg)\end{aligned}$$

in $\mathrm{Re}(s) > \alpha + \frac{1}{2}$*, where the sum is taken over all pairs of all positive integers* m *and* n*. It has an analytic continuation to the entire plane.*

(2) For a prime number p*, we have in* $\mathrm{Re}(s) > \alpha + \frac{1}{2}$

$$\zeta^{\alpha}_{p,p}(s) \cong \exp\left(\frac{2}{\pi i}\sum_{m\neq n}\frac{p^{-m(s-\frac{1}{2})-n(\alpha+\frac{1}{2})}}{m^2-n^2}\Big(\cosh(m\alpha\log p) + \frac{n}{m}\sinh(m\alpha\log p)\Big) \right. \tag{1.13}$$

$$\left. + \frac{1}{2\pi i}\left((\log p)(s-1-2\alpha)\log(1-p^{-s}) - \mathrm{Li}_2\left(p^{-s}\right) + \mathrm{Li}_2\left(p^{-s-2\alpha}\right)\right)\right), \tag{1.14}$$

which has an analytic continuation to the entire plane.

(3) The (p,∞)*-factor* $\zeta^{\alpha}_{p,\infty}(s)$ *of the double Riemann zeta function has an analytic continuation to the entire plane, and moreover* $\prod_p \zeta^{\alpha}_{p,\infty}(s)$ *has an analytic continuation to the entire plane with possible singularities at* $s = \frac{1}{2} - 2k \pm \alpha$ $(k \geq 0)$*,* $1-2k$ $(k\geq 0)$*,* $-2k$ $(k \geq 1)$*,* $\rho - 2k$ $(k \geq 0)$*,* $\frac{1}{2}+\rho\pm\alpha$*,* $\frac{3}{2}\pm\alpha$ *with* ρ *any nontrivial zero of* $\zeta(s)$*.*

(4) The (∞,∞)*-factor* $\zeta^{\alpha}_{\infty,\infty}(s)$ *of the double Riemann zeta function is analytic with possible singularities at* $s = -2n$*,* $-2n+\alpha+\frac{1}{2}$ *with* $n = 0, 1, 2, ...$

After completing the proof of (1)-(3) of Theorem 5 in §4-§6 of the text, we fix $\zeta^{\alpha}_{p,q}(s)$ as calculated there "with the integral constant 0"; in other words we fix $\zeta^{\alpha}_{p,q}(s)$ $(p \neq q)$ and $\zeta^{\alpha}_{p,p}(s)$ as in the right hand sides of (1) and (2) of Theorem 5; see §6 concerning $\zeta^{\alpha}_{p,\infty}(s)$ and $\prod_p \zeta^{\alpha}_{p,\infty}(s)$. In particular we will use these normalized Euler factors for the statement of Theorem 7 below.

Remark 1.4 To compare $\zeta^{\alpha}_{p,q}(s)$ $(p\neq q)$ and $\zeta^{\alpha}_{p,p}(s)$ with $\zeta_{p,q}(s)$ $(p \neq q)$ and $\zeta_{p,p}(s)$ treated in Theorem 1 and Theorem 2 respectively, it would be suggestive to take $\alpha = -1/2$ formally. For this parameter α, the sums over sinh-terms diverge. We look at the non-sinh parts:

$$\zeta^{-\frac{1}{2}}_{p,q}(s)_{\text{non-sinh}} = \exp\left(\frac{1}{\pi i}\sum_{m,n}\frac{\log p\log q}{(m\log p)^2-(n\log q)^2}\right.$$
$$\left.\left(\cosh\left(-\frac{m}{2}\log p\right)p^{-m(s-\frac{1}{2})} - \cosh\left(-\frac{n}{2}\log q\right)q^{-n(s-\frac{1}{2})}\right)\right)$$

and

$$\zeta^{-\frac{1}{2}}_{p,p}(s)_{\text{non-sinh}} = \exp\left(\frac{2}{\pi i}\sum_{m\neq n}\frac{\cosh(-\frac{m}{2}\log p)}{m^2-n^2}p^{-m(s-\frac{1}{2})}\right.$$
$$\left. + \frac{1}{2\pi i}\left((\log p)s\log(1-p^{-s}) - \mathrm{Li}_2(p^{-s}) + \mathrm{Li}_2(p^{-(s-1)})\right)\right).$$

Then, the elementary summation

$$\sum_{\substack{n=1\\n\neq\pm a}}^{\infty}\frac{1}{a^2-n^2}=\begin{cases}\dfrac{\pi}{2a}\cot(\pi a)-\dfrac{1}{2a^2} & (a\in\mathbf{C}\setminus\mathbf{Z})\\[2ex] -\dfrac{3}{4a^2} & (a\in\mathbf{Z}\setminus\{0\})\end{cases}$$

implies the followings:

$$\begin{aligned}\zeta_{p,q}^{-\frac{1}{2}}(s)_{\text{non-sinh}} \;=\; \exp\Bigg(& -\frac{i}{4}\sum_{m=1}^{\infty}\frac{\cot(\pi m\frac{\log p}{\log q})}{m}p^{-ms}-\frac{i}{4}\sum_{m=1}^{\infty}\frac{\cot(\pi m\frac{\log p}{\log q})}{m}p^{-m(s-1)}\\ & -\frac{i}{4}\sum_{n=1}^{\infty}\frac{\cot(\pi n\frac{\log q}{\log p})}{n}q^{-ns}-\frac{i}{4}\sum_{n=1}^{\infty}\frac{\cot(\pi n\frac{\log q}{\log p})}{n}q^{-n(s-1)}\\ & +\frac{i}{4\pi}\frac{\log q}{\log p}\left(\mathrm{Li}_2(p^{-s})+\mathrm{Li}_2(p^{-(s-1)})\right)+\frac{i}{4\pi}\frac{\log p}{\log q}\left(\mathrm{Li}_2(q^{-s})+\mathrm{Li}_2(q^{-(s-1)})\right)\Bigg)\end{aligned}$$

and

$$\zeta_{p,p}^{-\frac{1}{2}}(s)_{\text{non-sinh}} \;=\; \exp\left(\frac{5i}{4\pi}\mathrm{Li}_2(p^{-s})+\frac{i}{4\pi}\mathrm{Li}_2(p^{-(s-1)})-\frac{is\log p}{2\pi}\mathrm{Li}_1(p^{-s})\right).$$

Theorem 6 *The remaining factor $\zeta_0^{\alpha}(s)$ of the double Riemann zeta function is an analytic function on $\mathbf{C}$ with possible singularities at $s=\rho+\frac{1}{2}+\mathrm{sgn}(\mathrm{Im}(\rho))\alpha e^{i\theta}$ with $0\le\theta\le\pi$ for any nontrivial zero ρ of $\zeta(s)$ and at s belonging to $|s-1|\le 2\alpha$.*

In the next theorem we use the half Riemann zeta function $\zeta_+(s)$ studied in [HKW] (see §4 of the text) and the multiple gamma function $\Gamma_r(s)$ of Barnes [Bar] (see §2 of the text).

Theorem 7 *The Euler product for the double Riemann zeta function*

$$\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})\cong\left(\prod_{p,q}\zeta_{p,q}^{\alpha}(s)\right)\zeta_0^{\alpha}(s)\left(\frac{\prod_{m=1}^{\infty}\zeta_+(s+2m)}{\zeta_+(s-1)}\right)^2\Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2 s(s-2)$$

is absolutely convergent in $\mathrm{Re}(s)>\alpha+\frac{3}{2}$, where (p,q) runs through pairs of all (finite or infinite) places. It has an analytic continuation (with singularities) to the entire plane and satisfies a functional equation between s and $2-s$.

The construction of the later half of this paper is as follows. Theorem 4 is proved in §3. Then as an application of Theorem 4, Theorem 5 will be proved in §4-§7: $\zeta_{p,q}^{\alpha}(s)$ $(p\neq q)$ is treated in §4, $\zeta^{\alpha}_{p,p}(s)$ in §5, $\zeta^{\alpha}_{p,\infty}(s)$ in §6, and $\zeta^{\alpha}_{\infty,\infty}(s)$ in §7. Next, Theorem 6 is shown in §8. Consequently we prove Theorem 7 in §9. Lastly we briefly notice on remaining problems in §10.

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2 Proof of Theorems 1, 2 and 3

We recall the multiple sine functions studied in [KK2]. For this purpose we use the multiple Hurwitz zeta function investigated by Barnes [Bar] just 100 years ago:

$$\zeta_r(s,z,\boldsymbol{\omega}) = \sum_{n_1,\dots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s}$$

for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$. The definitions of the multiple gamma function and the multiple sine function are as follows:

$$\begin{aligned}\Gamma_r(z,\boldsymbol{\omega}) &= \exp\left(\frac{\partial}{\partial s}\zeta_r(s,z,\boldsymbol{\omega})\Big|_{s=0}\right) \\ &= \left(\prod_{\mathbf{n}\geq\mathbf{0}} (\mathbf{n}\cdot\boldsymbol{\omega}+z)\right)^{-1},\end{aligned}$$

$$\begin{aligned}S_r(z,\boldsymbol{\omega}) &= \Gamma_r(z,\boldsymbol{\omega})^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-z,\boldsymbol{\omega})^{(-1)^r} \\ &= \left(\prod_{\mathbf{n}\geq\mathbf{0}} (\mathbf{n}\cdot\boldsymbol{\omega}+z)\right)\left(\prod_{\mathbf{n}\geq\mathbf{1}} (\mathbf{n}\cdot\boldsymbol{\omega}-z)\right)^{(-1)^{r-1}}.\end{aligned}$$

We write $\Gamma_r(z) = \Gamma_r(z,(1,\dots,1))$ and $S_r(z) = S_r(z,(1,\dots,1))$ for simplicity. When $r=2$, we have $\boldsymbol{\omega} = (\omega_1,\omega_2)$ and

$$S_2(z,(\omega_1,\omega_2)) = \Gamma_2(z,(\omega_1,\omega_2))^{-1}\Gamma_2(\omega_1+\omega_2-z,(\omega_1,\omega_2)).$$

The study of this double sine function was originated by Shintani [Sh] towards the Kronecker's Jugendtraum for a real quadratic field. We obtained the following proposition in [KK2, Proposition 2.4].

Proposition 2.1 *We have an expression:*

$$S_r(z,\boldsymbol{\omega}) = e^{Q_{\boldsymbol{\omega}}(z)} z(z-|\boldsymbol{\omega}|)^{(-1)^{r-1}} {\prod_{\mathbf{n}\geq\mathbf{0}}}' P_r\left(-\frac{z}{\mathbf{n}\cdot\boldsymbol{\omega}}\right) P_r\left(\frac{z}{(\mathbf{n}+\mathbf{1})\cdot\boldsymbol{\omega}}\right)^{(-1)^{r-1}}$$

with $Q_{\boldsymbol{\omega}}(z)$ *a polynomial with* $\deg Q_{\boldsymbol{\omega}} \leq r$*,* $P_r(u) := (1-u)\exp(u+\frac{u^2}{2}+\cdots+\frac{u^r}{r})$ *and* $\mathbf{1} := (1,\cdots,1)$*.*

Lemma 2.2

$$S_2(z,\boldsymbol{\omega})S_2(-z,\boldsymbol{\omega}) = -4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right).$$

Proof. By the preceding proposition we have

$$\begin{aligned}
S_2(z,\boldsymbol{\omega})S_2(-z,\boldsymbol{\omega}) &\cong \frac{-z^2}{-z^2+(\omega_1+\omega_2)^2} \\
&\quad {\prod_{\mathbf{n}\geq\mathbf{0}}}' \frac{P_2\left(-\frac{z}{n_1\omega_1+n_2\omega_2}\right) P_2\left(\frac{z}{n_1\omega_1+n_2\omega_2}\right)}{P_2\left(\frac{z}{(n_1+1)\omega_1+(n_2+1)\omega_2}\right) P_2\left(-\frac{z}{(n_1+1)\omega_1+(n_2+1)\omega_2}\right)} \\
&= \frac{-z^2}{-z^2+(\omega_1+\omega_2)^2} P_2\left(\frac{z}{\omega_1+\omega_2}\right) P_2\left(-\frac{z}{\omega_1+\omega_2}\right) \\
&\quad \prod_{n_1=1}^{\infty} P_2\left(\frac{z}{n_1\omega_1}\right) P_2\left(-\frac{z}{n_1\omega_1}\right) \prod_{n_2=1}^{\infty} P_2\left(\frac{z}{n_2\omega_2}\right) P_2\left(-\frac{z}{n_2\omega_2}\right) \\
&\cong z^2 \prod_{n_1=1}^{\infty}\left(1-\frac{z}{n_1\omega_1}\right)\left(1+\frac{z}{n_1\omega_1}\right) \prod_{n_2=1}^{\infty}\left(1-\frac{z}{n_2\omega_2}\right)\left(1+\frac{z}{n_2\omega_2}\right) \\
&\cong \sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right).
\end{aligned}$$

Thus we put

$$S_2(z,\boldsymbol{\omega})S_2(-z,\boldsymbol{\omega}) = e^{Q(z)} 4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right) \tag{2.1}$$

and will first prove that the polynomial $Q(z)$ is a constant. The periodic property of $S_2(z,\boldsymbol{\omega})$ ([KK2, Theorem 2.1(a)])

$$S_2(z+\omega_1,\boldsymbol{\omega}) = S_2(z,\boldsymbol{\omega})S_1(z,\omega_2)^{-1} = S_2(z,\boldsymbol{\omega})\left(2\sin\frac{\pi z}{\omega_2}\right)^{-1}$$

shows that

$$S_2(-z-\omega_1,\boldsymbol{\omega}) = S_2(-z,\boldsymbol{\omega})S_1(-z-\omega_1,\omega_2) = S_2(-z,\boldsymbol{\omega})\left(-2\sin\frac{\pi(z+\omega_1)}{\omega_2}\right).$$

Substituting $z+\omega_1$ for z in (2.1), we have

$$S_2(z,\boldsymbol{\omega})\left(2\sin\frac{\pi z}{\omega_2}\right)^{-1} S_2(-z,\boldsymbol{\omega})\left(-2\sin\frac{\pi(z+\omega_1)}{\omega_2}\right) = -e^{Q(z+\omega_1)}4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi(z+\omega_1)}{\omega_2}\right),$$

namely

$$S_2(z,\boldsymbol{\omega})S_2(-z,\boldsymbol{\omega}) = e^{Q(z+\omega_1)}4\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right).$$

Thus we have $e^{Q(z)} = e^{Q(z+\omega_1)}$ for any z, which states that $Q(z)$ is a constant.

Next we put

$$S_2(z,\boldsymbol{\omega})S_2(-z,\boldsymbol{\omega}) = 4C\sin\left(\frac{\pi z}{\omega_1}\right)\sin\left(\frac{\pi z}{\omega_2}\right) \tag{2.2}$$

and will prove that the constant C is equal to -1. We consider the special value at $z=\omega_2/2$. First by [KK2, Remark 2.2],

$$S_2\left(\frac{\omega_2}{2},\boldsymbol{\omega}\right) = \sqrt{2}, \tag{2.3}$$

and again by the periodicity we have

$$S_2\left(-\frac{\omega_2}{2},\boldsymbol{\omega}\right) = S_2\left(\frac{\omega_2}{2},\boldsymbol{\omega}\right)S_1\left(-\frac{\omega_2}{2},\omega_1\right) = 2\sqrt{2}\sin\left(-\frac{\pi\omega_2}{2\omega_1}\right).$$

Hence (2.2) shows that $C=-1$. ∎

Remark 2.3 The formula (2.3) is an example of algebraic division values of multiple sine functions. Since the proof of (2.3) is omitted in [KK2, Remark 2.2] we give a proof here. By definition

$$S_2\left(\frac{\omega_2}{2},\boldsymbol{\omega}\right) = \frac{\Gamma_2(\frac{\omega_2}{2}+\omega_1,\boldsymbol{\omega})}{\Gamma_2(\frac{\omega_2}{2},\boldsymbol{\omega})}.$$

Here the periodicity of $\Gamma_2(x,\boldsymbol{\omega})$ in the form

$$\Gamma_2(x+\omega_1,\boldsymbol{\omega}) = \Gamma_2(x,\boldsymbol{\omega})\Gamma_1(x,\omega_2)^{-1}$$

gives

$$S_2\left(\frac{\omega_2}{2},\boldsymbol{\omega}\right) = \Gamma_1\left(\frac{\omega_2}{2},\omega_2\right)^{-1}.$$

Then the formula

$$\Gamma_1(x,\omega) = \frac{\Gamma(\frac{x}{\omega})}{\sqrt{2\pi}}\omega^{\frac{x}{\omega}-\frac{1}{2}}$$

originally due to Lerch implies (2.3).

Proof of Theorem 1. The statements (1)-(3) are proved in [KK1], since the main theorem of [KK1] shows that $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong \zeta_{p,q}(s)$ indeed. We recall that the main ingredient for the proof of (1)-(3) given in [KK1] is the following signed double Poisson summation formula.

Proposition 2.4 *Let $H(t)$ be an odd function in $L^1(\mathbf{R})$ with $H(t) = O(t^{-2})$ as $|t| \to \infty$, and put*

$$\widetilde{H}(u) = \int_{-\infty}^{\infty} H(t) e^{itu} dt.$$

Assume both a/b and b/a are generic and that the test function $H(t)$ satisfies

$$\widetilde{H}(x) = O(\mu^x) \tag{2.4}$$

as $x \to \infty$ for some $0 < \mu < 1$, then we have

$$\begin{aligned} &\sum_{m,n>0} H\left(2\pi\left(\frac{m}{a} + \frac{n}{b}\right)\right) + \frac{1}{2}\left(\sum_{m>0} H\left(2\pi\frac{m}{a}\right) + \sum_{n>0} H\left(2\pi\frac{n}{b}\right)\right) \\ &= -\frac{ia}{4\pi}\sum_{m>0} \cot\left(\pi\frac{ma}{b}\right)\widetilde{H}(ma) - \frac{ib}{4\pi}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)\widetilde{H}(nb) - \frac{iab}{8\pi^2}\widetilde{H}'(0). \end{aligned} \tag{2.5}$$

Here we say that a real number α is *generic* if and only if

$$\lim_{m\to\infty} \|m\alpha\|^{\frac{1}{m}} = 1,$$

where we put $\|x\| := \min\{|x - n| \ : \ n \in \mathbf{Z}\}$ for $x \in \mathbf{R}$. For example:

(1) If $\alpha \in (\overline{\mathbf{Q}} \cap \mathbf{R}) \setminus \mathbf{Q}$, then α is generic.

(2) Let $x, y \in \overline{\mathbf{Q}} \cap \mathbf{R}$. If $\alpha = \frac{\log x}{\log y} \notin \mathbf{Q}$, then α is transcendental and generic (Baker [B, Theorem 3.1], Baker-Wüstholz [BW]).

For our purpose we use the genericity of $\dfrac{\log p}{\log q}$ and $\dfrac{\log q}{\log p}$ for distinct primes p and q. We notice that this property is used even for the proof of convergence in Theorem 1(0): see [KK1].

Now we prove (4). By [KK1, Theorem 2], which is obtained via the above signed double Poisson summation formula, we have for $\mathrm{Re}(s) > 0$

$$
\begin{aligned}
&S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) \\
&= \exp\left(\frac{1}{2i}\sum_{m=1}^{\infty}\frac{1}{m}\cot\left(\pi m\frac{\log p}{\log q}\right)p^{-ms} + \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\log q}{\log p}\right)q^{-ns}\right. \\
&\quad + \frac{1}{2}\log\left(1-p^{-s}\right) + \frac{1}{2}\log\left(1-q^{-s}\right) \\
&\quad \left. + \frac{is^2(\log p)(\log q)}{8\pi} + \frac{1}{4}(\log p + \log q)s + \frac{\pi i}{12}\left(\frac{\log p}{\log q} + \frac{\log q}{\log p} + 3\right)\right) \\
&= \zeta_{p,q}(s)\exp\left(-\frac{is^2(\log p)(\log q)}{8\pi} + \frac{1}{4}(\log p + \log q)s + \frac{\pi i}{12}\left(\frac{\log p}{\log q} + \frac{\log q}{\log p} + 3\right)\right).
\end{aligned}
$$

As $\zeta_{p,q}(s)$ has a meromorphic continuation to all $s \in \mathbf{C}$, we also have

$$
\begin{aligned}
&S_2\left(-is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) \\
&\quad = \zeta_{p,q}(-s)\exp\left(-\frac{is^2(\log p)(\log q)}{8\pi} - \frac{1}{4}(\log p + \log q)s + \frac{\pi i}{12}\left(\frac{\log p}{\log q} + \frac{\log q}{\log p} + 3\right)\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\zeta_{p,q}(s)\zeta_{p,q}(-s) &= S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) S_2\left(-is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) \\
&\quad \exp\left(\frac{is^2(\log p)(\log q)}{4\pi} - \frac{\pi i}{6}\left(\frac{\log p}{\log q} + \frac{\log q}{\log p} + 3\right)\right) \\
&= S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) S_2\left(-is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) \\
&\quad \exp\left(\frac{i(\log p)(\log q)}{4\pi}\left(s^2 - \frac{2\pi^2}{3}\left(\frac{1}{(\log q)^2} + \frac{1}{(\log p)^2} + \frac{3}{(\log p)(\log q)}\right)\right)\right).
\end{aligned}
$$

Hence it suffices to show that

$$
S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) S_2\left(-is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right) = (p^{\frac{s}{2}} - p^{-\frac{s}{2}})(q^{\frac{s}{2}} - q^{-\frac{s}{2}}).
$$

This is proved in Lemma 2.2. ∎

Proof of Theorem 2. The idea of the proof is the same as the previous one. Properties (1)-(3) are shown in [KK1]; they easily follow from [KK2] also. Hence it is sufficient to prove (4).

We first have an expression

$$\zeta_{p,p}(s) = S_2\left(\frac{i\log p}{2\pi}s\right)\exp\left(\frac{i(\log p)^2}{8\pi}s^2 - \frac{\log p}{2}s - \frac{5\pi i}{12}\right).$$

Then we have

$$\zeta_{p,p}(s)\zeta_{p,p}(-s) = S_2\left(\frac{i\log p}{2\pi}s\right)S_2\left(-\frac{i\log p}{2\pi}s\right)\exp\left(\frac{i(\log p)^2}{4\pi}s^2 - \frac{5\pi i}{6}\right).$$

So using

$$S_2(x)S_2(-x) = (S_2(1+x)S_1(x))(S_2(1-x)S_1(-x)) = -S_1(x)^2,$$

we have

$$S_2\left(\frac{i\log p}{2\pi}s\right)S_2\left(-\frac{i\log p}{2\pi}s\right) = -S_1\left(\frac{i\log p}{2\pi}s\right)^2 = p^s(1-p^{-s})^2.$$

∎

Proof of Theorem 3. We use the following result of [KW2].

Proposition 2.5 *Let* $\zeta_r(s) = \prod_p \exp\left(\mathrm{Li}_r(p^{-s})\right)$ *for an integer* $r \geq 2$*, where* $\mathrm{Li}_r(s) = \sum_{n=1}^{\infty}\frac{u^n}{n^r}$ *is the polylogarithm of order* r*. Then* $\zeta_r(s)$ *is analytic in* $\mathrm{Re}(s) > 0$ *with the natural boundary* $\mathrm{Re}(s) = 0$*.*

Remark 2.6 $\zeta_1(s)$ is the Riemann zeta function.

Lemma 2.7 *The relation between* $Z(s)$ *and* $\zeta_2(s)$ *is given by*

$$\frac{Z'}{Z}(s) = -\frac{is}{2\pi}\left(\frac{\zeta_2'}{\zeta_2}\right)'(s) - \frac{\zeta'}{\zeta}(s).$$

Proof. From

$$\log\zeta_2(s) = \sum_p\sum_m \frac{1}{m^2}p^{-ms}$$

we obtain

$$\frac{\zeta_2'}{\zeta_2}(s) = -\sum_p\sum_m \frac{\log p}{m}p^{-ms}$$

and

$$\left(\frac{\zeta_2'}{\zeta_2}\right)'(s) = \sum_p\sum_m (\log p)^2 p^{-ms}.$$

On the other hand, from

$$\log Z(s) = \frac{i}{2\pi}\sum_p\sum_m \frac{1}{m^2}p^{-ms} - \sum_p\sum_m\left(1-\frac{i\log p}{2\pi}s\right)\frac{1}{m}p^{-ms}$$

we get

$$\begin{aligned}\frac{Z'}{Z}(s) &= -\frac{i}{2\pi}\sum_p\sum_m\frac{\log p}{m}p^{-ms}\\ &\quad +\frac{i}{2\pi}\sum_p\sum_m\frac{\log p}{m}p^{-ms}\\ &\quad +\sum_p\sum_m\left(1-\frac{i\log p}{2\pi}s\right)(\log p)p^{-ms}\\ &= \sum_p\sum_m\left(1-\frac{i\log p}{2\pi}s\right)(\log p)p^{-ms}.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{Z'}{Z}(s) &= -\frac{is}{2\pi}\sum_p\sum_m(\log p)^2p^{-ms} + \sum_p\sum_m(\log p)p^{-ms}\\ &= -\frac{is}{2\pi}\left(\frac{\zeta_2'}{\zeta_2}\right)'(s) - \frac{\zeta'}{\zeta}(s).\end{aligned}$$

∎

From this lemma and the result of [KW2], we see that $Z(s)$ is analytic in $\mathrm{Re}(s) > 0$ with the natural boundary $\mathrm{Re}(s) = 0$ since

$$\left(\frac{\zeta_2'}{\zeta_2}\right)'(s) = \sum_{m=1}^{\infty}\left(\sum_{n|m}\mu(n)n\right)\left(\frac{\zeta'}{\zeta}\right)'(ms)$$

has double poles at essential zeros of $\zeta(ms)$. The above equality is proved as follows. Let $\varphi(s) = \sum_p p^{-s}$. Then it is well-known that

$$\varphi(s) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\zeta(ns).$$

Hence

$$\begin{aligned}
\log \zeta_2(s) &= \sum_m \sum_p \frac{1}{m^2} p^{-ms} \\
&= \sum_m \frac{1}{m^2} \varphi(ms) \\
&= \sum_m \frac{1}{m^2} \sum_n \frac{\mu(n)}{n} \log \zeta(mns) \\
&= \sum_m \frac{1}{m^2} \left(\sum_{n|m} \mu(n) n \right) \log \zeta(ms).
\end{aligned}$$

Thus

$$\frac{\zeta_2'}{\zeta_2}(s) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{n|m} \mu(n) n \right) \frac{\zeta'}{\zeta}(ms)$$

and

$$\left(\frac{\zeta_2'}{\zeta_2} \right)'(s) = \sum_{m=1}^{\infty} \left(\sum_{n|m} \mu(n) n \right) \left(\frac{\zeta'}{\zeta} \right)'(ms).$$

∎

3 Signed Double Explicit Formula: Proof of Theorem 4

Lemma 3.1 *(1) Let M and N be distinct integers larger than 1. Then*

$$|\log M - \log N|^{-1} < 2\min(M, N) \leq 2\sqrt{MN}.$$

(2) For prime numbers p, q and positive integers m, n satisfying $p^m \neq q^n$ we have

$$|m \log p - n \log q|^{-1} < 2\min(p^m, q^n) \leq 2p^{m/2} q^{n/2}.$$

(3) Let M and N be distinct positive integers. Then

$$|\log M - \log N|^{-1} < \frac{\max(M, N)}{|M - N|}.$$

Proof. Since

$$\log \frac{M}{N} \geq \log \frac{N+1}{N} = \log\left(1+\frac{1}{N}\right)$$

for $M \geq N+1$ and

$$\log \frac{M}{N} \leq \log \frac{N-1}{N} = -\log\left(1+\frac{1}{N-1}\right)$$

for $M \leq N-1$, we have

$$\begin{aligned}
\left|\log \frac{M}{N}\right| &\geq \min\left\{\log\left(1+\frac{1}{N}\right),\ \log\left(1+\frac{1}{N-1}\right)\right\} \\
&= \log\left(1+\frac{1}{N}\right) \\
&= \int_0^{1/N} \frac{du}{1+u} \\
&> \int_0^{1/N} \frac{du}{2} \\
&= \frac{1}{2N}.
\end{aligned}$$

Hence

$$|\log M - \log N|^{-1} < 2N.$$

By symmetry we have also

$$|\log M - \log N|^{-1} < 2M.$$

This proves (1). Then (2) is the case $M = p^m$ and $N = q^n$ in (1).

Finally we show (3) in case of $M < N$:

$$\begin{aligned}
|\log M - \log N|^{-1} &= -\left(\log\left(1-\frac{N-M}{N}\right)\right)^{-1} \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{N-M}{N}\right)^k\right)^{-1} \\
&< \frac{N}{N-M},
\end{aligned}$$

by taking the term with $k=1$, since all other terms are positive. ∎

Proof of Theorem 4. Let D_T be the region defined by

$$D_T = \{s \in \mathbf{C} \mid |s| > \alpha,\ |\mathrm{Re}(s)| < \alpha,\ 0 < \mathrm{Im}(s) < T\}.$$

By Cauchy's theorem we have

$$\sum_{0<\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)<T} h(i\gamma_1+i\gamma_2) = \frac{1}{(2\pi i)^2}\int_{\partial D_T}\int_{\partial D_T} h(s_1+s_2)\frac{\xi'}{\xi}(s_1)\frac{\xi'}{\xi}(s_2)ds_1ds_2, \qquad (3.1)$$

where the integrals along ∂D_T are taken counter clockwise. Considering the limits as $T\to\infty$ in the both sides of (3.1), we have

$$\sum_{0<\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)} h(i\gamma_1+i\gamma_2) = \frac{1}{(2\pi i)^2}\int_{\partial D}\int_{\partial D} h(s_1+s_2)\frac{\xi'}{\xi}(s_1)\frac{\xi'}{\xi}(s_2)ds_1ds_2, \qquad (3.2)$$

where

$$D = \{s\in\mathbf{C}\mid |\mathrm{Re}(s)|<\alpha,\ |s|>\alpha,\ \mathrm{Im}(s)>0\}.$$

We decompose $\partial D = C_1\cup C_2\cup C_3$ with

$$\begin{aligned} C_1 &= \{s\in\partial D\mid \mathrm{Re}(s)=-\alpha\},\\ C_2 &= \{s\in\partial D\mid |s|=\alpha\},\\ C_3 &= \{s\in\partial D\mid \mathrm{Re}(s)=\alpha\}. \end{aligned}$$

We compute each double integral $I_{ij}=\frac{1}{(2\pi i)^2}\int_{C_i}\int_{C_j}$ in (3.2).

First we treat the integral along the vertical lines. We compute

$$\begin{aligned} I_{33} &= \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty h(2\alpha+i(t_1+t_2))\frac{\xi'}{\xi}(\alpha+it_1)\frac{\xi'}{\xi}(\alpha+it_2)dt_1dt_2\\ &= \frac{1}{4\pi^2}\int_0^\infty h(2\alpha+it)\int_0^t\frac{\xi'}{\xi}(\alpha+it_1)\frac{\xi'}{\xi}(\alpha+i(t-t_1))dt_1dt \end{aligned}$$

and

$$\begin{aligned} I_{11} &= \frac{1}{4\pi^2}\int_\infty^0\int_\infty^0 h(-2\alpha+i(t_1+t_2))\frac{\xi'}{\xi}(-\alpha+it_1)\frac{\xi'}{\xi}(-\alpha+it_2)dt_1dt_2\\ &= \frac{1}{4\pi^2}\int_{-\infty}^0 h(2\alpha+it)\int_0^t\frac{\xi'}{\xi}(\alpha+it_1)\frac{\xi'}{\xi}(\alpha+i(t-t_1))dt_1dt. \end{aligned}$$

Hence

$$
\begin{aligned}
I_{11}+I_{33} &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} h(2\alpha+it)\int_0^t \frac{\xi'}{\xi}(\alpha+it_1)\frac{\xi'}{\xi}(\alpha+i(t-t_1))dt_1dt & (3.3)\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} h(2\alpha+it) \\
&\quad \left(\int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right)\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+i(t-t_1)\right)dt_1\right. & (3.4)\\
&\quad +\int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right)\frac{\zeta'}{\zeta}\left(\alpha+\frac{1}{2}+i(t-t_1)\right)dt_1 & (3.5)\\
&\quad +\int_0^t \frac{\zeta'}{\zeta}\left(\alpha+\frac{1}{2}+it_1\right)\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+i(t-t_1)\right)dt_1 & (3.6)\\
&\quad \left.+\int_0^t \frac{\zeta'}{\zeta}\left(\alpha+\frac{1}{2}+it_1\right)\frac{\zeta'}{\zeta}\left(\alpha+\frac{1}{2}+i(t-t_1)\right)dt_1\right)dt. & (3.7)
\end{aligned}
$$

The contribution from (3.4) is equal to (1.8). Next we compute that both (3.5) and (3.6) are equal to

$$
-\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(\alpha+\frac{1}{2})}}p^{-imt}\int_0^t p^{imt'}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it'\right)dt'.
$$

Thus we have (1.6).

The contribution from (3.7) is calculated as

$$
\frac{1}{4\pi^2}\sum_{p,q,m,n}\frac{\log p\log q}{p^{m(\alpha+\frac{1}{2})}q^{n(\alpha+\frac{1}{2})}}\int_{-\infty}^{\infty} H_\alpha(t)q^{-nit}\int_0^t p^{-mit_1}q^{nit_1}dt_1dt.
$$

When p,q,m,n satisfy that $p^m\neq q^n$, the integral on t_1 is equal to $\dfrac{i(p^{-mit}q^{nit}-1)}{m\log p-n\log q}$, and the summand is

$$
\frac{i\log p\log q}{p^{m(\alpha+\frac{1}{2})}q^{n(\alpha+\frac{1}{2})}(m\log p-n\log q)}\int_{-\infty}^{\infty} H_\alpha(t)\left(p^{-mit}-q^{-nit}\right)dt. \tag{3.8}
$$

In case $p^m=q^n$, which means $p=q$ and $m=n$, the integral on t_1 is equal to t, and the summand is

$$
\frac{(\log p)^2}{p^{2m(\alpha+\frac{1}{2})}}\int_{-\infty}^{\infty} tH_\alpha(t)p^{-mit}dt. \tag{3.9}
$$

Thus the total contribution from (3.7) to (3.3) is

$$\frac{i}{4\pi^2}\sum_{\substack{p,q,m,n\\ p^m\neq q^n}}\frac{\log p\log q}{\log(\frac{p^m}{q^n})p^{m(\alpha+\frac12)}q^{n(\alpha+\frac12)}}\left(\widetilde{H_\alpha}(-m\log p)-\widetilde{H_\alpha}(-n\log q)\right)$$

$$+\frac{1}{4\pi^2}\sum_{p,m}\frac{(\log p)^2}{p^{2m(\alpha+\frac12)}}\widetilde{tH_\alpha(t)}(-m\log p). \quad (3.10)$$

Thus we obtain (1.2), (1.4) and (1.5). Here we notice that the absolute convergence of (1.2) follows from Lemma 3.1.

We calculate I_{13} and I_{31} to have

$$\begin{aligned} I_{13}+I_{31} &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}h(it)\int_0^t\frac{\xi'}{\xi}(\alpha+it_1)\frac{\xi'}{\xi}(\alpha-i(t-t_1))dt_1dt \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}h(it) \\ &\quad\left(\int_0^t\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12+it_1\right)\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12-i(t-t_1)\right)dt_1\right. && (3.11)\\ &\quad+\int_0^t\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12+it_1\right)\frac{\zeta'}{\zeta}\left(\alpha+\frac12-i(t-t_1)\right)dt_1 && (3.12)\\ &\quad+\int_0^t\frac{\zeta'}{\zeta}\left(\alpha+\frac12+it_1\right)\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12-i(t-t_1)\right)dt_1 && (3.13)\\ &\quad\left.+\int_0^t\frac{\zeta'}{\zeta}\left(\alpha+\frac12+it_1\right)\frac{\zeta'}{\zeta}\left(\alpha+\frac12-i(t-t_1)\right)dt_1\right)dt. && (3.14)\end{aligned}$$

We find that (3.11) is equal to (1.9). Next (3.12) and (3.13) are computed as

$$(3.12)=-\frac{1}{4\pi^2}\sum_{m=1}^{\infty}\sum_p\frac{\log p}{p^{m(\alpha+\frac12)}}\int_{-\infty}^{\infty}H_0(t)\int_0^t p^{im(t-t')}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12+it'\right)dt'dt$$

and

$$(3.13)=-\frac{1}{4\pi^2}\sum_{m=1}^{\infty}\sum_p\frac{\log p}{p^{m(\alpha+\frac12)}}\int_{-\infty}^{\infty}H_0(t)\int_0^t p^{-im(t-t')}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac12-it'\right)dt'dt.$$

Thus (3.12) + (3.13) is equal to (1.7). Finally by taking into account that $H_0(t)$ is an odd function, we have (1.1) and (1.3) from (3.14).

We write the remaining integrals as $I_2+I_2'-I_{22}$ with $I_2:=I_{21}+I_{22}+I_{23}$ and $I_2':=I_{12}+I_{22}+I_{32}$, and first calculate I_2:

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{\partial D}h(s_1+s_2)\frac{\xi'}{\xi}(s_1)ds_1\right)\frac{\xi'}{\xi}(s_2)ds_2 \\ &= \frac{1}{2\pi i}\int_{C_2}\sum_{\gamma_1}h(i\gamma_1+s_2)\frac{\xi'}{\xi}(s_2)ds_2, \end{aligned}$$

where γ_1 runs through the nontrivial zeros of $\zeta(s)$ with $\mathrm{Re}(\gamma_1) > 0$ and $|\mathrm{Im}(\gamma_1)| < \alpha$. Putting $s_2 = \alpha e^{i\theta}$, this together with I_2' makes the term (1.10). Finally the integral I_{22} becomes (1.11).

∎

In the next corollary we split (1.3) into partial sums over $m \neq n$ and $m = n$, and combine the former with (1.1). We also combine (1.2) and (1.4). Put $\alpha = \frac{1}{2} + \delta$ with $\delta > 0$.

Corollary 1 *For $\delta > 0$ it holds that*

$$
\begin{aligned}
&\sum_{\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)>0} H_0(\gamma_1+\gamma_2) \\
= \ & \frac{i}{4\pi^2}\sum_{\substack{p,q,m,n\\ p^m\neq q^n}} \frac{\log p \log q}{\log(p^m q^n)} \frac{1}{p^{m(1+\delta)}q^{n(1+\delta)}} \left(\widetilde{H_0}(-m\log p) + \widetilde{H_0}(-n\log q)\right) && (3.15)\\
+ \ & \frac{i}{4\pi^2}\sum_{p}\sum_{m=1}^{\infty} \frac{\log p}{mp^{2m(1+\delta)}}\widetilde{H_0}(-m\log p) && (3.16)\\
+ \ & \frac{i}{4\pi^2}\sum_{\substack{p,q,m,n\\ p^m\neq q^n}} \frac{\log p \log q}{\log(\frac{p^m}{q^n})} \frac{1}{p^{m(1+\delta)}q^{n(1+\delta)}} \left(\widetilde{H_{\frac{1+2\delta}{2}}}(-m\log p) - \widetilde{H_{\frac{1+2\delta}{2}}}(-n\log q)\right) && (3.17)\\
+ \ & \frac{1}{4\pi^2}\sum_{p}(\log p)^2 \sum_{m=1}^{\infty} p^{-2m(1+\delta)}(t\widetilde{H_{\frac{1+2\delta}{2}}(t)})(-m\log p) && (3.18)\\
- \ & \frac{1}{2\pi^2}\sum_{p}\sum_{m=1}^{\infty}\frac{\log p}{p^{m(1+\delta)}}\int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t)\int_0^t p^{im(t'-t)}\frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta+it')dt'dt && (3.19)\\
- \ & \frac{1}{2\pi^2}\sum_{p}\sum_{m=1}^{\infty}\frac{\log p}{p^{m(1+\delta)}}\int_{-\infty}^{\infty} H_0(t)\int_0^t \mathrm{Re}\left(p^{im(t-t')}\frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta+it')\right)dt'dt && (3.20)\\
+ \ & \frac{1}{4\pi^2}\int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t)\int_0^t \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta+it_1)\frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta+i(t-t_1))dt_1dt && (3.21)\\
+ \ & \frac{1}{4\pi^2}\int_{-\infty}^{\infty} H_0(t)\int_0^t \frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta+it_1)\frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(1+\delta-i(t-t_1))dt_1dt && (3.22)\\
- \ & \frac{1+2\delta}{2\pi}\int_0^{\pi}\sum_{\mathrm{Re}(\gamma_1)>0} h(i\gamma_1+(\tfrac{1}{2}+\delta)e^{i\theta})\frac{\xi'}{\xi}((\tfrac{1}{2}+\delta)e^{i\theta})e^{i\theta}d\theta && (3.23)\\
- \ & \frac{(1+2\delta)^2}{16\pi^2}\int_0^{\pi}\int_0^{\pi} h((\tfrac{1}{2}+\delta)e^{i\theta_1}+(\tfrac{1}{2}+\delta)e^{i\theta_2})\frac{\xi'}{\xi}((\tfrac{1}{2}+\delta)e^{i\theta_1})\frac{\xi'}{\xi}((\tfrac{1}{2}+\delta)e^{i\theta_2})e^{i(\theta_1+\theta_2)}d\theta_1d\theta_2, && (3.24)
\end{aligned}
$$

where the sum in the left hand side is taken over pairs $(\frac{1}{2}+i\gamma_1,\ \frac{1}{2}+i\gamma_2)$ of nontrivial zeros of the Riemann zeta function, p, q denote prime numbers, and $m, n \in \mathbf{Z}$, $m, n \geq 1$.

4 $\zeta_{p,q}^{\alpha}(s)$: Proof of Theorem 5(1)

The goal of this section is to calculate the factor $\zeta_{p,q}^{\alpha}(s)$ of the double Riemann zeta function for distinct prime numbers p and q. The case $p = q$ and those involving infinite places will be treated in §5-§7.

We assume $\mathrm{Re}(s) > 1 + 2\delta$ for $\delta = \alpha - \frac{1}{2}$ and take the test function $h(t)$ as in (1.12). Then

$$\begin{aligned} H_{\frac{1+2\delta}{2}}(t) &= \frac{1}{(it+1+2\delta+s)^2} - \frac{1}{(it+1+2\delta-s)^2} \\ &= \frac{-1}{(t-i(1+2\delta+s))^2} + \frac{1}{(t-i(1+2\delta-s))^2}, \end{aligned} \tag{4.1}$$

and we compute for $x < 0$ that

$$\begin{aligned} \widetilde{H_{\frac{1+2\delta}{2}}}(x) &= \int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t) e^{ixt} dt \\ &= -2\pi i \mathrm{Res}_{t=(1+2\delta-s)i}(H_{\frac{1+2\delta}{2}}(t)e^{ixt}) \\ &= 2\pi x e^{-x(1+2\delta-s)}, \end{aligned}$$

where we considered the integral along the lower half circle since $|e^{ixt}| = e^{-x\mathrm{Im}t}$ and $x < 0$.

Lemma 4.1 *Let $1/2 < \alpha < 1$. For distinct prime numbers p and q, the (p,q)-Euler factor of $\hat{\zeta}(s+\frac{1}{2},\mathbf{Z}) \otimes \hat{\zeta}(s+\frac{1}{2},\mathbf{Z})$ is given by*

$$\exp\left(\frac{1}{\pi i}\sum_{m,\ n} \frac{(\log p)(\log q)p^{-m/2}q^{-n/2}}{(m\log p)^2-(n\log q)^2}\right.$$
$$\left.\left(\frac{\cosh(m\alpha\log p)}{p^{ms}q^{n\alpha}} + \frac{n\log q}{m\log p}\frac{\sinh(m\alpha\log p)}{p^{ms}q^{n\alpha}} - \frac{m\log p}{n\log q}\frac{\sinh(n\alpha\log q)}{p^{m\alpha}q^{ns}} - \frac{\cosh(n\alpha\log q)}{p^{m\alpha}q^{ns}}\right)\right)$$

where the sum is taken over all pairs of positive integers m and n, and is absolutely convergent for $\mathrm{Re}(s) > \alpha - \frac{1}{2}$.

Proof. Put

$$L(s) = \prod_{\substack{\rho_1,\rho_2 \\ \mathrm{Im}(\rho_1),\mathrm{Im}(\rho_2)>0}} \left(1 - \frac{s}{\rho_1+\rho_2}\right)\exp\left(\frac{s}{\rho_1+\rho_2} + \frac{s^2}{2(\rho_1+\rho_2)^2}\right),$$

where ρ_1 and ρ_2 run through the upper half zeros of $\xi(s) = \hat{\zeta}(s+\frac{1}{2})$. Then we have by definition

$$(\xi\otimes\xi)(s) \cong \frac{L(s)}{L(-s)}\zeta_+(s)^{-2}\zeta_+(s+1)^{-2}(s-1)s^2(s+1) \tag{4.2}$$

with

$$\zeta_+(s) = \prod_{\substack{\rho:\hat{\zeta}(\rho)=0 \\ \mathrm{Im}(\rho)>0}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

being the half Riemann zeta function introduced in [HKW]. In fact, from

$$\xi(s) \cong \frac{Z^1(s)}{Z^0(s)Z^2(s)}$$

with

$$\begin{aligned} Z^0(s) &= s + \frac{1}{2}, \\ Z^1(s) &= \prod_{\hat{\zeta}(\rho)=0} \left(s - \left(\rho - \frac{1}{2}\right)\right), \\ Z^2(s) &= s - \frac{1}{2} \end{aligned}$$

we have

$$\begin{aligned} \xi(s) \otimes \xi(s) \cong \big(Z^1(s) \otimes Z^1(s)\big) \times \big(Z^2(s) \otimes Z^1(s)\big)^{-2} \times \big(Z^0(s) \otimes Z^1(s)\big)^{-2} \\ \times \big(Z^2(s) \otimes Z^2(s)\big) \times \big(Z^2(s) \otimes Z^0(s)\big)^2 \times \big(Z^0(s) \otimes Z^0(s)\big) \end{aligned}$$

with

$$\begin{aligned} Z^1(s) \otimes Z^1(s) &\cong \frac{L(s)}{L(-s)}, \\ Z^2(s) \otimes Z^1(s) &\cong \zeta_+(s), \\ Z^0(s) \otimes Z^1(s) &\cong \zeta_+(s+1), \\ Z^2(s) \otimes Z^2(s) &\cong s - 1, \\ Z^2(s) \otimes Z^0(s) &\cong s, \\ Z^0(s) \otimes Z^0(s) &\cong s + 1. \end{aligned}$$

If we put $(\xi \otimes \xi)(s) = e^{F(s)}$ and

$$M(s) := \zeta_+(s)^{-2}\zeta_+(s+1)^{-2}(s-1)s^2(s+1),$$

it holds that

$$F''(s) = \frac{d}{ds}\left(\frac{L'}{L}(-s) + \frac{L'}{L}(s) + \frac{M'}{M}(s)\right).$$

In other words we have

$$\frac{d}{ds}\left(\frac{L'}{L}(-s) + \frac{L'}{L}(s)\right) = F''(s) - \frac{d}{ds}\frac{M'}{M}(s). \tag{4.3}$$

Then the left hand side of Corollary 1 is equal to that of (4.3). Thus it is equal to the right hand side of Corollary 1, among which the terms (3.15) and (3.17) produce the desired factors in the theorem. Now we compute that

$$
\begin{aligned}
&(3.15) + (3.17) \\
&= \frac{i}{4\pi^2} \sum_{\substack{p,m,q,n \\ p^m \neq q^n}} \frac{(\log p)(\log q)}{p^{m(\frac{1}{2}+\alpha)} q^{n(\frac{1}{2}+\alpha)}} \left(\frac{\widetilde{H_\alpha}(-m\log p) - \widetilde{H_\alpha}(-n\log q)}{m\log p - n\log q} + \frac{\widetilde{H_0}(-m\log p) + \widetilde{H_0}(-n\log q)}{m\log p + n\log q} \right).
\end{aligned}
$$

The terms in the big parentheses are equal to

$$
\begin{aligned}
\frac{1}{(m\log p)^2 - (n\log q)^2} \quad \times \quad & \Big(\Big(\widetilde{H_\alpha}(-m\log p) + \widetilde{H_0}(-m\log p)\Big)\, m\log p \\
& + \Big(\widetilde{H_\alpha}(-m\log p) - \widetilde{H_0}(-m\log p)\Big)\, n\log q \\
& - \Big(\widetilde{H_\alpha}(-n\log q) - \widetilde{H_0}(-n\log q)\Big)\, m\log p \\
& - \Big(\widetilde{H_\alpha}(-n\log q) + \widetilde{H_0}(-n\log q)\Big)\, n\log q \Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(3.15) + (3.17) \quad &= \quad \frac{1}{2\pi i} \sum_{\substack{p,m,q,n \\ p^m \neq q^n}} \frac{(\log p)(\log q) p^{-m(\frac{1}{2}+\alpha)} q^{-n(\frac{1}{2}+\alpha)}}{(m\log p)^2 - (n\log q)^2} \\
& \Big(m^2 (\log p)^2 p^{-ms}(p^{2m\alpha} + 1) + m(\log p) n(\log q) p^{-ms}(p^{2m\alpha} - 1) \\
& - m(\log p) n(\log q) q^{-ns}(q^{2n\alpha} - 1) - n^2(\log q)^2 q^{-ns}(q^{2n\alpha} + 1) \Big) \\
&= \quad \frac{1}{\pi i} \sum_{\substack{p,q,m,n \\ p^m \neq q^n}} \frac{(\log p)(\log q) p^{-m/2} q^{-n/2}}{(m\log p)^2 - (n\log q)^2} \\
& \Big(m^2 (\log p)^2 \frac{\cosh(m\alpha\log p)}{p^{ms} q^{n\alpha}} + m(\log p) n(\log q) \frac{\sinh(m\alpha\log p)}{p^{ms} q^{n\alpha}} \\
& - m(\log p) n(\log q) \frac{\sinh(n\alpha\log q)}{p^{m\alpha} q^{ns}} - n^2(\log q)^2 \frac{\cosh(n\alpha\log q)}{p^{m\alpha} q^{ns}} \Big).
\end{aligned}
$$

Integrating twice leads to

$$
\begin{aligned}
& \frac{1}{\pi i} \sum_{\substack{p,q,m,n \\ p^m \neq q^n}} \frac{(\log p)(\log q) p^{-m/2} q^{-n/2}}{(m\log p)^2 - (n\log q)^2} \\
& \left(\frac{\cosh(m\alpha\log p)}{p^{ms} q^{n\alpha}} + \frac{n\log q}{m\log p} \frac{\sinh(m\alpha\log p)}{p^{ms} q^{n\alpha}} - \frac{m\log p}{n\log q} \frac{\sinh(n\alpha\log q)}{p^{m\alpha} q^{ns}} - \frac{\cosh(n\alpha\log q)}{p^{m\alpha} q^{ns}} \right) \qquad (4.4)
\end{aligned}
$$

which is a factor of the double zeta function. ∎

If we shift the variable as $s \mapsto s-1$ in Lemma 4.1 we get the factor in Theorem 5(1).

Now we show the analytic continuation of $\zeta^{\alpha}_{p,q}(s)$ for all $s \in \mathbf{C}$. For $\mathrm{Re}(s) > 0$ and $\beta > 0$, let

$$\varphi_{p,q}(s,\beta) = \sum_{m,n} \frac{\log p \log q}{(m \log p)^2 - (n \log q)^2} p^{-ms} q^{-n\beta},$$

and

$$\psi_{p,q}(s,\beta) = \sum_{m,n} \frac{n}{m} \frac{(\log q)^2}{(m \log p)^2 - (n \log q)^2} p^{-ms} q^{-n\beta}.$$

Then we have

$$\begin{aligned}
\zeta^{\alpha}_{p,q}(s) &= \exp\Bigg(\frac{1}{2\pi i} \bigg(\varphi_{p,q}\left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) + \varphi_{p,q}\left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) \\
&\quad -\psi_{p,q}\left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) + \psi_{p,q}\left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) + \varphi_{q,p}\left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) \\
&\quad +\varphi_{q,p}\left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) - \psi_{q,p}\left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) + \psi_{q,p}\left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right) \bigg) \Bigg).
\end{aligned}$$

Hence it is sufficient to show that $\varphi_{p,q}(s,\beta)$ and $\psi_{p,q}(s,\beta)$ are analytic in $s \in \mathbf{C}$.

For a generic $\alpha \in \mathbf{R}$ and any $\beta > 0$, we put

$$L_{p,q}(s;\alpha,\beta) = \sum_{m,n} \frac{p^{-ms} q^{-n\beta}}{m - n\alpha}$$

in $\mathrm{Re}(s) > 0$. The absolute convergence is easily seen from the genericity of α. Moreover, the integral expression

$$L_{p,q}(s;\alpha,\beta) = \int_0^1 \frac{dv}{(p^s - v)(q^{\beta} v^{\alpha} - 1)}$$

gives the analytic continuation to all $s \in \mathbf{C}$. This equality is shown as follows:

$$\begin{aligned}
\int_0^1 \frac{dv}{(p^s - v)(q^{\beta} v^{\alpha} - 1)} &= \int_0^1 \left(\sum_{m=1}^{\infty} p^{-ms} v^{m-1} \right) \left(\sum_{n=1}^{\infty} q^{-n\beta} v^{-n\alpha} \right) dv \\
&= \sum_{m,n} p^{-ms} q^{-n\beta} \int_0^1 v^{m-n\alpha-1} dv \\
&= \sum_{m,n} \frac{p^{-ms} q^{-n\beta}}{m - n\alpha}.
\end{aligned}$$

Hence if we see that

$$\varphi'_{p,q}(s,\beta) = -\frac{\log q}{2}\left(L_{p,q}\left(s;\frac{\log q}{\log p},\beta\right) + L_{p,q}\left(s;-\frac{\log q}{\log p},\beta\right)\right)$$

and

$$\psi'_{p,q}(s,\beta) = -\frac{\log q}{2}\left(L_{p,q}\left(s;\frac{\log q}{\log p},\beta\right) - L_{p,q}\left(s;-\frac{\log q}{\log p},\beta\right)\right),$$

then we would have the analytic continuation of $\varphi_{p,q}(s,\beta)$ and $\psi_{p,q}(s,\beta)$ for all $s \in \mathbf{C}$. To prove these identities notice first that:

$$\begin{aligned} L_{p,q}(s;\alpha,\beta) + L_{p,q}(s;-\alpha,\beta) &= \sum_{m,n}\left(\frac{1}{m-n\alpha} + \frac{1}{m+n\alpha}\right)p^{-ms}q^{-n\beta} \\ &= \sum_{m,n}\frac{2m}{m^2-n^2\alpha^2}p^{-ms}q^{-n\beta} \end{aligned}$$

and

$$\begin{aligned} L_{p,q}(s;\alpha,\beta) - L_{p,q}(s;-\alpha,\beta) &= \sum_{m,n}\left(\frac{1}{m-n\alpha} - \frac{1}{m+n\alpha}\right)p^{-ms}q^{-n\beta} \\ &= \sum_{m,n}\frac{2n\alpha}{m^2-n^2\alpha^2}p^{-ms}q^{-n\beta}. \end{aligned}$$

Thus

$$\begin{aligned} \varphi'_{p,q}(s,\beta) &= -\sum_{m,n}\frac{m(\log p)^2\log q}{(m\log p)^2-(n\log q)^2}p^{-ms}q^{-n\beta} \\ &= -\frac{\log q}{2}\sum_{m,n}\frac{2m}{m^2-n^2(\frac{\log q}{\log p})^2}p^{-ms}q^{-n\beta} \\ &= -\frac{\log q}{2}\left(L_{p,q}\left(s;\frac{\log q}{\log p},\beta\right) + L_{p,q}\left(s;-\frac{\log q}{\log p},\beta\right)\right) \end{aligned}$$

and

$$\begin{aligned} \psi'_{p,q}(s,\beta) &= -\sum_{m,n}\frac{n(\log q)^2\log p}{(m\log p)^2-(n\log q)^2}p^{-ms}q^{-n\beta} \\ &= -\frac{\log q}{2}\sum_{m,n}\frac{2n(\frac{\log q}{\log p})}{m^2-n^2(\frac{\log q}{\log p})^2}p^{-ms}q^{-n\beta} \\ &= -\frac{\log q}{2}\left(L_{p,q}\left(s;\frac{\log q}{\log p},\beta\right) - L_{p,q}\left(s;-\frac{\log q}{\log p},\beta\right)\right). \end{aligned}$$

This completes the proof of the analytic continuation of $\zeta^{\alpha}_{p,q}(s)$.

5 $\zeta^{\alpha}_{p,p}(s)$: Proof of Theorem 5(2)

The (p,p)-factors are the sum of the contributions of four terms (3.15)-(3.18) in case of $p=q$. For (3.15) and (3.17), we compute in the same way as in the previous theorem to get by putting $p=q$

$$\exp\left(\frac{1}{\pi i}\sum_{m\neq n}\frac{p^{-(m+n)/2}}{m^2-n^2}\right.$$
$$\left.\left(\frac{\cosh(m\alpha\log p)}{p^{ms+n\alpha}}+\frac{n}{m}\frac{\sinh(m\alpha\log p)}{p^{ms+n\alpha}}-\frac{m}{n}\frac{\sinh(n\alpha\log p)}{p^{m\alpha+ns}}-\frac{\cosh(n\alpha\log p)}{p^{m\alpha+ns}}\right)\right)$$
$$=\exp\left(\frac{2}{\pi i}\sum_{m\neq n}\frac{p^{-m(\frac{1}{2}+s)-n(\frac{1}{2}+\alpha)}}{m^2-n^2}\left(\cosh(m\alpha\log p)+\frac{n}{m}\sinh(m\alpha\log p)\right)\right).$$

The shift $s\mapsto s-1$ gives the first line (1.13).

Next we calculate the contributions from (3.16) and (3.18). By taking the test function (4.1), we compute

$$(3.16)=\frac{1}{2\pi i}\sum_p\sum_{m=1}^{\infty}(\log p)^2p^{-m(s+2(1+\delta))}.$$

Thus

$$\iint(3.16)dsds=\frac{1}{2\pi i}\sum_p\mathrm{Li}_2\left(p^{-(s+1+2\alpha)}\right).$$

The shift $s\mapsto s-1$ leads to the last term in (1.14).

Next the calculation for (3.18) shows

$$(3.18)=\frac{1}{2\pi i}\sum_p(\log p)^2\sum_{m=1}^{\infty}(1+(s-1-2\delta)(-m\log p))p^{-m(s+1)},$$

where we compute

$$\widetilde{H}'_{\alpha}(x)=it\widetilde{H_{\alpha}}(x)=2\pi(1+(s-2\alpha)x)e^{x(s-2\alpha)}.$$

By partial integration

$$\begin{aligned}\int(3.18)ds &= \frac{1}{2\pi i}\sum_p(\log p)^2\sum_{m=1}^{\infty}\left(\left(\frac{1}{-m\log p}+s-1-2\delta\right)p^{-m(s+1)}-\int p^{-m(s+1)}ds\right)\\ &= \frac{1}{2\pi i}\sum_p(\log p)^2\sum_{m=1}^{\infty}(s-1-2\delta)\,p^{-m(s+1)}.\end{aligned}$$

Hence

$$\begin{aligned}\iint (3.18) ds ds &= \frac{1}{2\pi i}\sum_p (\log p)^2 \sum_{m=1}^{\infty}\left(\frac{s-1-2\delta}{-m\log p}p^{-m(s+1)} - \int \frac{1}{-m\log p}p^{-m(s+1)}ds\right)\\ &= \frac{1}{2\pi i}\sum_p \sum_{m=1}^{\infty}\left(-\frac{\log p}{m}p^{-m(s+1)}(s-1-2\delta) - \frac{1}{m^2}p^{-m(s+1)}\right)\\ &= \frac{1}{2\pi i}\sum_p \left((\log p)\left(\log\left(1-p^{-(s+1)}\right)\right)(s-1-2\delta) - \mathrm{Li}_2\left(p^{-(s+1)}\right)\right).\end{aligned}$$

Thus we determined the form of $\zeta_{p,p}^{\alpha}(s)$. The analytic continuation of $\zeta_{p,p}^{\alpha}(s)$ is given as follows. We decompose $\zeta_{p,p}^{\alpha}(s)$ into two parts

$$\zeta_{p,p}^{\alpha}(s) = Z_1(s)Z_2(s)$$

with

$$Z_1(s) = \exp\left(\frac{2}{\pi i}\sum_{m\neq n}\frac{p^{-m(s-\frac{1}{2})-n(\alpha+\frac{1}{2})}}{m^2-n^2}\left(\cosh(m\alpha\log p) + \frac{n}{m}\sinh(m\alpha\log p)\right)\right)$$

and

$$Z_2(s) = \exp\left(\frac{1}{2\pi i}\left((s-1-2\alpha)\log p\log(1-p^{-s}) - \mathrm{Li}_2(p^{-s}) + \mathrm{Li}_2(p^{-s-2\alpha})\right)\right).$$

We start with $Z_2(s)$ since it is treated directly by Theorem 2. In fact,

$$\exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(p^{-s})\right) = \zeta_{p,p}(s)\exp\left(-\left(1+\frac{\log p}{2\pi i}s\right)\log(1-p^{-s})\right)$$

and

$$\exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(p^{-s-2\alpha})\right) = \zeta_{p,p}(s+2\alpha)^{-1}\exp\left(\left(1+\frac{\log p}{2\pi i}(s+2\alpha)\right)\log(1-p^{-s-2\alpha})\right)$$

imply that

$$\begin{aligned}Z_2(s) = \zeta_{p,p}(s)\zeta_{p,p}(s+2\alpha)^{-1}\exp\Bigg(&-\left(1+\frac{\log p}{2\pi i}(1+2\alpha)\right)\log(1-p^{-s})\\ &+\left(1+\frac{\log p}{2\pi i}(s+2\alpha)\right)\log(1-p^{-s-2\alpha})\Bigg),\end{aligned}$$

so $Z_2(s)$ is analytic in $s\in\mathbf{C}$ from Theorem 2(1).

Next we look at $Z_1(s)$. For $\mathrm{Re}(s) > 0$ and $\beta > 0$, let

$$\varphi_p(s,\beta) = \sum_{m \neq n} \frac{p^{-ms-n\beta}}{m^2 - n^2}$$

and

$$\psi_p(s,\beta) = \sum_{m \neq n} \frac{n}{m} \frac{p^{-ms-n\beta}}{m^2 - n^2}.$$

Then we have

$$\begin{aligned} Z_1(s) &= \exp\left(\frac{2}{\pi i} \sum_{m \neq n} \frac{p^{-m(s-\frac{1}{2})-n(\alpha+\frac{1}{2})}}{m^2-n^2} \left(\frac{p^{m\alpha}+p^{-m\alpha}}{2} + \frac{n}{m} \frac{p^{m\alpha}-p^{-m\alpha}}{2} \right) \right) \\ &= \exp\Bigg(\frac{1}{\pi i} \left(\varphi_p \left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2} \right) + \varphi_p \left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2} \right) \right. \\ &\qquad \left. + \psi_p \left(s-\alpha-\frac{1}{2}, \alpha+\frac{1}{2} \right) - \psi_p \left(s+\alpha-\frac{1}{2}, \alpha+\frac{1}{2} \right) \right) \Bigg). \end{aligned}$$

Hence it is sufficient to show the analyticity of $\varphi_p(s,\beta)$ and $\psi_p(s,\beta)$ in $s \in \mathbf{C}$. Now, putting

$$L_p^-(s,\beta) = \sum_{m \neq n} \frac{p^{-ms-n\beta}}{m-n}$$

and

$$L_p^+(s,\beta) = \sum_{m \neq n} \frac{p^{-ms-n\beta}}{m+n}$$

we see that

$$\begin{aligned} \varphi_p'(s,\beta) &= \sum_{m \neq n} \frac{-m(\log p) p^{-ms-n\beta}}{m^2-n^2} \\ &= -\frac{\log p}{2} \left(L_p^-(s,\beta) + L_p^+(s,\beta) \right) \end{aligned}$$

and

$$\begin{aligned} \psi_p'(s,\beta) &= \sum_{m \neq n} \frac{-n(\log p) p^{-ms-n\beta}}{m^2-n^2} \\ &= -\frac{\log p}{2} \left(L_p^-(s,\beta) - L_p^+(s,\beta) \right). \end{aligned}$$

Thus our task is to show that $L_p^-(s,\beta)$ and $L_p^+(s,\beta)$ have analytic continuations to all $s \in \mathbf{C}$. We show the needed analyticity by giving the following expressions:

$$L_p^-(s,\beta) = \frac{1}{1-p^{s+\beta}} \log\left(\frac{1-p^{-s}}{1-p^{-\beta}}\right)$$

and

$$L_p^+(s,\beta) = \frac{1}{2}\log(1-p^{-s-\beta}) + \frac{1}{1-p^{\beta-s}}\log(1-p^{-s}) + \frac{1}{1-p^{s-\beta}}\log(1-p^{-\beta}).$$

These expressions are obtained by direct calculations:

$$\begin{aligned}
L_p^-(s,\beta) &= \sum_{m>n} \frac{p^{-ms-n\beta}}{m-n} + \sum_{m<n} \frac{p^{-ms-n\beta}}{m-n} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{p^{-ks-n(s+\beta)}}{k} - \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{p^{-m(s+\beta)-l\beta}}{l} \\
&= -\log(1-p^{-s})\frac{p^{-(s+\beta)}}{1-p^{-(s+\beta)}} + \log(1-p^{-\beta})\frac{p^{-(s+\beta)}}{1-p^{-(s+\beta)}} \\
&= \frac{1}{1-p^{s+\beta}} \log\left(\frac{1-p^{-s}}{1+p^{-\beta}}\right).
\end{aligned}$$

and

$$\begin{aligned}
L_p^+(s,\beta) &= \sum_{m,n\geq 1} \frac{p^{-ms-n\beta}}{m+n} - \sum_{m=1}^{\infty} \frac{p^{-m(s+\beta)}}{2m} \\
&= \sum_{l=2}^{\infty} \frac{1}{l}\left(p^{-(l-1)s-\beta} + \cdots + p^{-s-(l-1)\beta}\right) - \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}p^{-m(s+\beta)} \\
&= \sum_{l=2}^{\infty} \frac{1}{l}p^{-s-\beta}\frac{p^{-(l-1)s}-p^{-(l-1)\beta}}{p^{-s}-p^{-\beta}} + \frac{1}{2}\log(1-p^{-s-\beta}) \\
&= \sum_{l=1}^{\infty} \frac{1}{l}p^{-s-\beta}\frac{p^{-(l-1)s}-p^{-(l-1)\beta}}{p^{-s}-p^{-\beta}} + \frac{1}{2}\log(1-p^{-s-\beta}) \\
&= -\frac{p^{-\beta}}{p^{-s}-p^{-\beta}}\log(1-p^{-s}) + \frac{p^{-s}}{p^{-s}-p^{-\beta}}\log(1-p^{-\beta}) + \frac{1}{2}\log(1-p^{-s-\beta}) \\
&= \frac{1}{2}\log(1-p^{-s-\beta}) + \frac{1}{1-p^{\beta-s}}\log(1-p^{-s}) + \frac{1}{1-p^{s-\beta}}\log(1-p^{-\beta}).
\end{aligned}$$

Thus we proved the analyticity of $\zeta_{p,p}^{\alpha}(s)$ for all $s \in \mathbf{C}$.∎

Remark 5.1 We must distinguish $L_p^-(s,\beta)$ from "$L_{p,p}(s;1,\beta)$" carefully. In fact "$L_{p,p}(s;1,\beta)$" diverges.

6 $\zeta^{\alpha}_{p,\infty}(s)$: Proof of Theorem 5(3)

We calculate $\zeta^{\alpha}_{p,\infty}(s)$, and moreover we describe the analyticity and possible singularities of $\prod_p \zeta^{\alpha}_{p,\infty}(s)$.

We first deal with (3.19) by using the formula

$$\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s) = -\frac{1}{2}\log\pi - \frac{\gamma}{2} - \frac{1}{s} - \sum_{k=1}^{\infty}\left(\frac{1}{s+2k} - \frac{1}{2k}\right)$$

with γ the Euler constant. We begin with the calculation of the integral on t' in (3.19):

$$\begin{aligned}&\int_0^t p^{imt'}\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(1+\delta+it')dt' \\ &\quad = \int_0^t p^{imt'}\left(-\frac{1}{2}\log\pi - \frac{\gamma}{2} - \frac{1}{1+\delta+it'} - \sum_{k=1}^{\infty}\left(\frac{1}{1+\delta+it'+2k} - \frac{1}{2k}\right)\right)dt'. \quad (6.1)\end{aligned}$$

We divide the integral (6.1) into three parts. First we compute

$$\begin{aligned}\int_0^t p^{imt'}\left(-\frac{1}{2}\log\pi - \frac{\gamma}{2}\right)dt' &= \left(-\frac{1}{2}\log\pi - \frac{\gamma}{2}\right)\frac{p^{imt}-1}{im\log p} \\ &= i\left(\frac{1}{2}\log\pi + \frac{\gamma}{2}\right)\frac{p^{imt}-1}{m\log p}. \quad (6.2)\end{aligned}$$

Secondly we have by putting $t'' = 1+\delta+it'$

$$\begin{aligned}-\int_0^t \frac{p^{imt'}}{1+\delta+it'}dt' &= -\int_{1+\delta}^{it+1+\delta}\frac{p^{m(t''-1-\delta)}}{t''}\frac{dt''}{i} \\ &= i\int_{1+\delta}^{it+1+\delta}\sum_{n=0}^{\infty}a_n(t'')^{n-1}dt'' \\ &= i\left(a_0\log\frac{it+1+\delta}{1+\delta} + \sum_{n=0}^{\infty}a_{n+1}\frac{(it+1+\delta)^{n+1}-(1+\delta)^{n+1}}{n+1}\right) \quad (6.3)\end{aligned}$$

with the expansion $p^{m(t''-1-\delta)} = \sum_{n=0}^{\infty}a_n(t'')^n$ with $a_n = p^{m(-1-\delta)}(m\log p)^n/n!$.

Next we calculate by putting $t'' = 1+\delta+it'+2k$

$$
\begin{aligned}
&-\int_0^t p^{imt'} \sum_{k=1}^{\infty} \left(\frac{1}{1+\delta+it'+2k} - \frac{1}{2k} \right) dt' \\
&= -\sum_{k=1}^{\infty} p^{-2mk} \int_{1+\delta+2k}^{it+1+\delta+2k} p^{m(t''-1-\delta)} \left(\frac{1}{t''} - \frac{1}{2k} \right) \frac{dt''}{i} \\
&= i \sum_{k=1}^{\infty} p^{-2mk} \sum_{n=0}^{\infty} a_n \int_{1+\delta+2k}^{it+1+\delta+2k} \left((t'')^{n-1} - \frac{(t'')^n}{2k} \right) dt'' \\
&= i \sum_{k=1}^{\infty} p^{-2mk} \left(a_0 \log \frac{it+1+\delta+2k}{1+\delta+2k} + \sum_{n=0}^{\infty} \left(a_{n+1} - \frac{a_n}{2k} \right) \int_{1+\delta+2k}^{it+1+\delta+2k} (t'')^n dt'' \right) \\
&= i \sum_{k=1}^{\infty} p^{-2mk} \left(a_0 \log \frac{it+1+\delta+2k}{1+\delta+2k} + \sum_{n=0}^{\infty} \left(a_{n+1} - \frac{a_n}{2k} \right) \frac{(it+1+\delta+2k)^{n+1} - (1+\delta+2k)^{n+1}}{n+1} \right).
\end{aligned}
\tag{6.4}
$$

By (6.2), (6.3), (6.4), we deduce that (3.19) is equal to

$$
\begin{aligned}
&-\frac{i}{2\pi^2} \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{m(1+\delta)}} \int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t) p^{-imt} \left(\left(\frac{1}{2}\log \pi + \frac{\gamma}{2} \right) \frac{p^{imt}-1}{m \log p} \right. \\
&\left. + \sum_{k=0}^{\infty} p^{-2mk} \left(a_0 \log \frac{it+1+\delta+2k}{1+\delta+2k} + \sum_{n=0}^{\infty} a_{n,k} \frac{(it+1+\delta+2k)^{n+1} - (1+\delta+2k)^{n+1}}{n+1} \right) \right) dt
\end{aligned}
$$

with $a_{n,0} = a_{n+1}$ and $a_{n,k} = a_{n+1} - \frac{a_n}{2k}$ for $k \geq 1$. For avoiding the singular point $t = i(1+\delta+2k)$ in the log term, we calculate the integral by integrating along the lower half circle. It encircles the double pole $t=-i(s-1-2\delta)$ of (4.1). Thus (3.19) is equal to

$$\frac{1}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(1+\delta)}}\mathrm{Res}_{t=-i(s-1-2\delta)}\frac{p^{-imt}}{(t+i(s-1-2\delta))^2}\left(\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)\frac{p^{imt}-1}{m\log p}\right.$$
$$\left.+\sum_{k=0}^{\infty}p^{-2mk}\left(a_0\log\frac{it+1+\delta+2k}{1+\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}\frac{(it+1+\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\right)\right)$$
$$=\frac{1}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(1+\delta)}}\left.\frac{d}{dt}\right|_{t=-i(s-1-2\delta)}p^{-imt}\left(\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)\frac{p^{imt}-1}{m\log p}\right.$$
$$\left.+\sum_{k=0}^{\infty}p^{-2mk}\left(a_0\log\frac{it+1+\delta+2k}{1+\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}\frac{(it+1+\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\right)\right)$$
$$=\frac{1}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(s-\delta)}}\left(-im(\log p)\left(\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)\frac{p^{m(s-1-2\delta)}-1}{m\log p}\right.\right.$$
$$\left.+\sum_{k=0}^{\infty}p^{-2mk}\left(a_0\log\frac{s-\delta+2k}{1+\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}\frac{(s-\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\right)\right)$$
$$\left.+i\left(\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)p^{m(s-1-2\delta)}+\sum_{k=0}^{\infty}p^{-2mk}\left(\frac{a_0}{s-\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}(s-\delta+2k)^n\right)\right)\right)$$
$$=\frac{1}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(s-\delta)}}\left(i\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)-im(\log p)a_0\sum_{k=0}^{\infty}p^{-2mk}\log\frac{s-\delta+2k}{1+\delta+2k}\right.\tag{6.5}$$
$$-im(\log p)\sum_{k=0}^{\infty}p^{-2mk}\sum_{n=0}^{\infty}a_{n,k}\frac{(s-\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\tag{6.6}$$
$$\left.+i\sum_{k=0}^{\infty}p^{-2mk}\left(\frac{a_0}{s-\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}(s-\delta+2k)^n\right)\right).\tag{6.7}$$

We compute the sum over p and m in (6.5)-(6.7). Since

$$\frac{\zeta'}{\zeta}(s)=-\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{ms}}$$

and

$$\left(\frac{\zeta'}{\zeta}\right)'(s)=\sum_p\sum_{m=1}^{\infty}\frac{m(\log p)^2}{p^{ms}},$$

(6.5) is equal to

$$-\frac{i}{\pi}\left(\left(\frac{1}{2}\log\pi+\frac{\gamma}{2}\right)\frac{\zeta'}{\zeta}(s-\delta)+\sum_{k=0}^{\infty}\left(\frac{\zeta'}{\zeta}\right)'(s+2k+1)\log\frac{s-\delta+2k}{1+\delta+2k}\right).\tag{6.8}$$

As we also have

$$\frac{d^r}{ds^r}\frac{\zeta'}{\zeta}(s) = (-1)^{r-1}\sum_p\sum_{m=1}^{\infty}\frac{m^r(\log p)^{r+1}}{p^{ms}},$$

(6.6) is equal to

$$\begin{aligned}
&\frac{-i}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{m(\log p)^2}{p^{m(s-\delta)}}\sum_{k=0}^{\infty}p^{-2mk}\sum_{n=0}^{\infty}\left(a_{n+1}-\frac{a_n}{2k}\right)\frac{(s-\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\\
&\quad = \quad \frac{-i}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(s-\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\\
&\qquad\left(\frac{1}{(n+1)!}\sum_p\sum_{m=1}^{\infty}\frac{m^{n+2}(\log p)^{n+3}}{p^{m(s+2k+1)}}-\frac{1}{2k(n!)}\sum_p\sum_{m=1}^{\infty}\frac{m^{n+1}(\log p)^{n+2}}{p^{m(s+2k+1)}}\right)\\
&\quad = \quad \frac{-i}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(s-\delta+2k)^{n+1}-(1+\delta+2k)^{n+1}}{n+1}\\
&\qquad\left(\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{\zeta'}{\zeta}\right)^{(n+2)}(s+2k+1)-\frac{(-1)^n}{2k(n!)}\left(\frac{\zeta'}{\zeta}\right)^{(n+1)}(s+2k+1)\right),
\end{aligned}\tag{6.9}$$

where we used the convention that $a_n/2k = 0$ if $k = 0$. Finally (6.7) is

$$\begin{aligned}
&\frac{i}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(s-\delta)}}\sum_{k=0}^{\infty}p^{-2mk}\left(\frac{a_0}{s-\delta+2k}+\sum_{n=0}^{\infty}a_{n,k}(s-\delta+2k)^n\right)\\
&\quad = \quad \frac{i}{\pi}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(s+1)}}\sum_{k=0}^{\infty}p^{-2mk}\left(\frac{1}{s-\delta+2k}+\sum_{n=0}^{\infty}\left(\frac{(m\log p)^{n+1}}{(n+1)!}-\frac{(m\log p)^n}{2k(n!)}\right)(s-\delta+2k)^n\right)\\
&\quad = \quad \frac{i}{\pi}\sum_{k=0}^{\infty}\left(\frac{1}{s-\delta+2k}\sum_p\sum_{m=1}^{\infty}\frac{\log p}{p^{m(s+2k+1)}}\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}(s-\delta+2k)^n\left(\frac{1}{(n+1)!}\sum_p\sum_{m=1}^{\infty}\frac{m^{n+1}(\log p)^{n+2}}{p^{m(s+2k+1)}}-\frac{1}{2k(n!)}\sum_p\sum_{m=1}^{\infty}\frac{m^n(\log p)^{n+1}}{p^{m(s+2k+1)}}\right)\right)\\
&\quad = \quad \frac{i}{\pi}\sum_{k=0}^{\infty}\left(\frac{-1}{s-\delta+2k}\frac{\zeta'}{\zeta}(s+2k+1)\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}(s-\delta+2k)^n\left(\frac{(-1)^n}{(n+1)!}\left(\frac{\zeta'}{\zeta}\right)^{(n+1)}(s+2k+1)-\frac{(-1)^{n-1}}{2k(n!)}\left(\frac{\zeta'}{\zeta}\right)^{(n)}(s+2k+1)\right)\right)
\end{aligned}\tag{6.10}$$

Integrating (6.8)-(6.10) twice gives a function in s which is analytic except at $s = \delta - 2k$ $(k \geq 0)$, $-2k$ $(k \geq 0)$, $-1-2k$ $(k \geq 1)$, $-1+\rho-2k$ $(k \geq 0)$, $\delta+\rho$ and $\delta+1$. Writing in terms of $\alpha = \frac{1}{2}+\delta$ and the shift $s \mapsto s-1$ lead to possible singularities at $s = \frac{1}{2} - 2k + \alpha$ $(k \geq 0)$, $1-2k$ $(k \geq 0)$, $-2k$ $(k \geq 1)$, $\rho - 2k$ $(k \geq 0)$, $\frac{1}{2}+\rho+\alpha$ and $\frac{3}{2}+\alpha$.

Treating (3.20) similarly we have singularities of $\prod_p \zeta_{p,\infty}^{\alpha}(s)$ at $s = \frac{1}{2} - 2k - \alpha$, $\frac{1}{2}+\rho-\alpha$ and $\frac{3}{2}-\alpha$.

7 $\zeta_{\infty,\infty}^{\alpha}(s)$: Proof of Theorem 5(4)

Lemma 7.1 *We have the following formula on an indefinite integral:*

$$\int \frac{\log x}{ax+b}dx = \begin{cases} \frac{1}{a}\left(\log\left(1+\frac{a}{b}x\right)\log x + \mathrm{Li}_2\left(-\frac{a}{b}x\right)\right) + C & \left(|x| < \left|\frac{b}{a}\right|\right) \\ \frac{1}{a}\left(\log\left(1+\frac{b}{ax}\right)\log x - b\mathrm{Li}_2\left(-\frac{b}{ax}\right) + \frac{(\log x)^2}{2}\right) + C, & \left(|x| > \left|\frac{b}{a}\right|\right) \end{cases}$$

where a and b are nonzero complex numbers. In particular it is an analytic function in x except at $x = 0$, $-b/a$.

Proof. We first note that for $|x| < 1$

$$\frac{d}{dx}\mathrm{Li}_2(x) = \frac{d}{dx}\sum_{k=1}^{\infty}\frac{x^k}{k^2} = \sum_{k=1}^{\infty}\frac{x^{k-1}}{k} = \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k} = -\frac{\log(1-x)}{x}.$$

Hence when $|x| < |\frac{b}{a}|$, by differentiating the right hand side of Lemma,

$$\begin{aligned} \frac{d}{dx}(\text{R.H.S}) &= \frac{1}{a}\left(\frac{a/b}{1+\frac{a}{b}x}\log x + \log\left(1+\frac{a}{b}x\right)\frac{1}{x} - \frac{\log(1+\frac{a}{b}x)}{-\frac{a}{b}x}\left(-\frac{a}{b}\right)\right) \\ &= \frac{\log x}{ax+b}. \end{aligned}$$

We similarly compute for the case $|x| > |\frac{b}{a}|$. ∎

Put

$$\varepsilon_n = \begin{cases} \frac{1}{2n} & (n = 1,2,3,...) \\ -\frac{\log \pi}{2} - \frac{\gamma}{2} & (n = 0) \end{cases}$$

and we use the expansion

$$\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s) = -\sum_{k=0}^{\infty}\left(\frac{1}{s+2k} - \varepsilon_k\right).$$

The integral on t_1 in (3.21) is

$$\int_0^t \sum_{n=0}^{\infty}\left(\frac{1}{2n+1+\delta+it_1}-\varepsilon_n\right)\sum_{m=0}^{\infty}\left(\frac{1}{2m+1+\delta+i(t-t_1)}-\varepsilon_m\right)dt_1.$$

The quadratic term in t_1 is calculated as follows:

$$\begin{aligned}
&\int_0^t \frac{1}{2n+1+\delta+it_1}\frac{1}{2m+1+\delta+i(t-t_1)}dt_1\\
&= \frac{1}{2(m+n+1+\delta)+it}\int_0^t\left(\frac{1}{2n+1+\delta+it_1}+\frac{1}{2m+1+\delta+i(t-t_1)}\right)dt_1\\
&= \frac{-i}{2(m+n+1+\delta)+it}\left(\log\frac{2n+1+\delta+it}{2n+1+\delta}-\log\frac{2m+1+\delta}{2m+1+\delta+it}\right)\\
&= \frac{-i}{2(m+n+1+\delta)+it}\log\left(\left(1+\frac{it}{2n+1+\delta}\right)\left(1+\frac{it}{2m+1+\delta}\right)\right).
\end{aligned}$$

Its contribution to (3.21) is

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}H_{\frac{1+2\delta}{2}}(t)\sum_{m,n=0}^{\infty}\frac{-i}{2(m+n+1+\delta)+it}\log\left(\left(1+\frac{it}{2n+1+\delta}\right)\left(1+\frac{it}{2m+1+\delta}\right)\right)dt\\
&= \frac{-i}{2\pi}\mathrm{Res}_{t=-i(s-1-2\delta)}\frac{1}{(t-i(1+2\delta-s))^2}\\
&\quad\sum_{m,n=0}^{\infty}\frac{-i}{2(m+n+1+\delta)+it}\log\left(\left(1+\frac{it}{2n+1+\delta}\right)\left(1+\frac{it}{2m+1+\delta}\right)\right)\\
&= \frac{-1}{2\pi}\left.\frac{d}{dt}\right|_{t=-i(s-1-2\delta)}\sum_{m,n=0}^{\infty}\frac{1}{2(m+n+1+\delta)+it}\log\left(\left(1+\frac{it}{2n+1+\delta}\right)\left(1+\frac{it}{2m+1+\delta}\right)\right)\\
&= \frac{-1}{2\pi}\sum_{m,n=0}^{\infty}\left(\frac{-i}{(2(m+n+1+\delta)+it)^2}\log\left(\left(1+\frac{it}{2n+1+\delta}\right)\left(1+\frac{it}{2m+1+\delta}\right)\right)\right.\\
&\quad\left.\left.+\frac{1}{2(m+n+1+\delta)+it}\left(\frac{i}{2n+1+\delta+it}+\frac{i}{2m+1+\delta+it}\right)\right)\right|_{t=-i(s-1-2\delta)}\\
&= \frac{-1}{2\pi}\sum_{m,n=0}^{\infty}\left(\frac{-i}{(2(m+n)+1+s)^2}\log\left(\left(1+\frac{s-1-2\delta}{2n+1+\delta}\right)\left(1+\frac{s-1-2\delta}{2m+1+\delta}\right)\right)\right.\\
&\quad\left.+\frac{1}{2(m+n)+1+s}\left(\frac{i}{s+2n-\delta}+\frac{i}{s+2m-\delta}\right)\right)\\
&= \frac{-i}{2\pi}\sum_{m,n=0}^{\infty}\frac{d}{ds}\left(\frac{1}{2(m+n)+1+s}\log\left(\left(1+\frac{s-1-2\delta}{2n+1+\delta}\right)\left(1+\frac{s-1-2\delta}{2m+1+\delta}\right)\right)\right).
\end{aligned}$$

Thus

$$\int (\text{the quadratic part in } (3.21)) ds$$

$$= \frac{-i}{2\pi} \sum_{m,n=0}^{\infty} \frac{1}{2(m+n)+1+s} \log \left(\left(1 + \frac{s-1-2\delta}{2n+1+\delta} \right) \left(1 + \frac{s-1-2\delta}{2m+1+\delta} \right) \right).$$

By putting $x = 1 + \frac{s-1-2\delta}{2n+1+\delta}$, it follows that

$$\frac{1}{2(m+n)+1+s} \log \left(1 + \frac{s-1-2\delta}{2n+1+\delta} \right) = \frac{\log x}{ax+b} \tag{7.1}$$

with $a = 2n+1+\delta$ and $b = 2m+1+\delta$. By lemma 7.1 its indefinite integral is analytic in s except at $s = -2n+\delta$ and $s = -2n-1$. The shift $s \mapsto s-1$ leads us to the singularities at $s = -2n+\delta+1$ and $s = -2n$.

Next we deal with non-quadratic parts in (3.21). Their contributions to (3.21) are

$$\begin{aligned}
&\frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t) \int_0^t \sum_{m,n=0}^{\infty} \left(\frac{-\varepsilon_n}{2m+1+\delta+i(t-t_1)} + \frac{-\varepsilon_m}{2n+1+\delta+it_1} + \varepsilon_m \varepsilon_n \right) dt_1 dt \\
= \; & \frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_{\frac{1+2\delta}{2}}(t) \sum_{m,n=0}^{\infty} \Bigg(-i\varepsilon_n \log \frac{2m+1+\delta}{2m+1+\delta+it} \\
& \qquad\qquad + i\varepsilon_m \log \frac{2n+1+\delta+it}{2n+1+\delta} + \varepsilon_m \varepsilon_n t \Bigg) dt \\
= \; & \frac{-i}{2\pi} \mathrm{Res}_{t=-i(s-1-2\delta)} \frac{1}{(t-i(1+2\delta-s))^2} \sum_{m,n=0}^{\infty} \Bigg(i\varepsilon_n \log \frac{2m+1+\delta+it}{2m+1+\delta} \\
& \qquad\qquad + i\varepsilon_m \log \frac{2n+1+\delta+it}{2n+1+\delta} + \varepsilon_m \varepsilon_n t \Bigg) \\
= \; & \frac{-i}{2\pi} \left. \frac{d}{dt} \right|_{t=-i(s-1-2\delta)} \sum_{m,n=0}^{\infty} \Bigg(i\varepsilon_n \log \frac{2m+1+\delta+it}{2m+1+\delta} \\
& \qquad\qquad + i\varepsilon_m \log \frac{2n+1+\delta+it}{2n+1+\delta} + \varepsilon_m \varepsilon_n t \Bigg) \\
= \; & \frac{-i}{2\pi} \sum_{m,n=0}^{\infty} \left. \left(\frac{-\varepsilon_n}{2m+1+\delta+it} + \frac{-\varepsilon_m}{2n+1+\delta+it} + \varepsilon_m \varepsilon_n \right) \right|_{t=-i(s-1-2\delta)} \\
= \; & \frac{-i}{2\pi} \sum_{m,n=0}^{\infty} \left(\frac{-\varepsilon_n}{2m-\delta+s} + \frac{-\varepsilon_m}{2n-\delta+s} + \varepsilon_m \varepsilon_n \right).
\end{aligned}$$

By integrating twice, we see that the resulting function is analytic except at $s = -2n+\delta$ with $n = 0, 1, 2, \ldots$

This completes the proof of Theorem 5(4) and thus Theorem 5.

8 $\zeta_0^\alpha(s)$: Proof of Theorem 6

We first deal with (1.10):

$$-\frac{\alpha}{\pi}\sum_{\mathrm{Re}(\gamma)>0}\int_0^\pi\left(\frac{1}{(i\gamma+\alpha e^{i\theta}+s)^2}-\frac{1}{(i\gamma+\alpha e^{i\theta}-s)^2}\right)\frac{\xi'}{\xi}(\alpha e^{i\theta})e^{i\theta}d\theta.$$

This function in s is holomorphic in $\mathrm{Re}(s)>\frac{1}{2}+\alpha$, and it has an analytic continuation to all $s\in\mathbf{C}$ except for possible singularities on $\pm C(\gamma)$ with

$$C(\gamma)=\{i\gamma+\alpha e^{i\theta}\mid 0\le\theta\le\pi\}.$$

Next we deal with (1.11), which is equal to

$$-\frac{\alpha^2}{4\pi^2}\int_0^\pi\int_0^\pi\left(\frac{1}{(\alpha e^{i\theta_1}+\alpha e^{i\theta_2}+s)^2}-\frac{1}{(\alpha e^{i\theta_1}+\alpha e^{i\theta_2}-s)^2}\right)\frac{\xi'}{\xi}(\alpha e^{i\theta_1})\frac{\xi'}{\xi}(\alpha e^{i\theta_2})e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2.$$

The integrand is a bounded function in θ_1 and θ_2 for any fixed s, and thus the integral defines an analytic function for all $s\in\mathbf{C}$ except for possible singularities in $|s|\le 2\alpha$.

The shift $s\mapsto s-1$ leads to the desired result.

9 $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$: Proof of Theorem 7

We first prove the absolute convergence of the multiple Euler product. Then other properties follow from our previous discussions noting that

$$\hat{\zeta}(s,\mathbf{Z})\otimes\hat{\zeta}(s,\mathbf{Z})=(\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z}))(\zeta(s,\mathbf{Z})\otimes\Gamma_{\mathbf{R}}(s))^2(\Gamma_{\mathbf{R}}(s)\otimes\Gamma_{\mathbf{R}}(s)).$$

We see the convergence from the following lemma.

Lemma 9.1 *Let $\sigma>\alpha+\frac{3}{2}$. Then:*

(1)

$$\sum_{p\neq q}\sum_{m,n\ge 1}\frac{(\log p)(\log q)}{|(m\log p)^2-(n\log q)^2|}\frac{p^{m\alpha}}{q^{n(\alpha+\frac{1}{2})}}p^{-m(\sigma-\frac{1}{2})}<\infty.$$

(2)

$$\sum_{p\neq q}\sum_{m,n\ge 1}\frac{n}{m}(\log q)^2\frac{1}{|(m\log p)^2-(n\log q)^2|}\frac{p^{m\alpha}}{q^{n(\alpha+\frac{1}{2})}}p^{-m(\sigma-\frac{1}{2})}<\infty.$$

In the proof we use the following lemma.

Lemma 9.2 *Let $a > 0$ and $w > 0$. Then*

$$\sum_{l=1}^{\infty} \frac{1}{l}(a+l)^{-w} < 2^{-w} a^{-w/2} \frac{w+2}{w}.$$

Proof.

$$\begin{aligned}
\sum_{l=1}^{\infty} \frac{1}{l}(a+l)^{-w} &< \sum_{l=1}^{\infty} \frac{1}{l}\left(2\sqrt{al}\right)^{-w} \\
&= 2^{-w} a^{-w/2} \sum_{l=1}^{\infty} l^{-1-\frac{w}{2}} \\
&= 2^{-w} a^{-w/2} \zeta\left(1+\frac{w}{2}\right) \\
&< 2^{-w} a^{-w/2} \frac{1+\frac{w}{2}}{\frac{w}{2}} \\
&= 2^{-w} a^{-w/2} \frac{w+2}{w}.
\end{aligned}$$

Here we used the inequality $\zeta(s) < s/(s-1)$ for $s > 1$. ∎

Proof of Lemma 9.1. We first prove (1). From Lemma 3.1(3) and the fact that

$$|m \log p + n \log q| > 1,$$

It suffices to prove for $\sigma > \alpha + \frac{3}{2}$ that

$$A := \sum_{\substack{p,q,m,n \\ p^m > q^n}} \frac{p^m}{p^m - q^n} q^{-n(\alpha+\frac{1}{2})} p^{-m(\sigma-\alpha-\frac{1}{2})} < \infty \tag{9.1}$$

and that

$$B := \sum_{\substack{p,q,m,n \\ p^m < q^n}} \frac{q^n}{q^n - p^m} q^{-n(\alpha+\frac{1}{2})} p^{-m(\sigma-\alpha-\frac{1}{2})} < \infty, \tag{9.2}$$

since the factors $\log p$, $\log q$, m and n are easily treated by differentiation (or directly). For proving (9.1) we have by putting $l = p^m - q^n \geq 1$ and by Lemma 9.2

$$\begin{aligned}
\sum_{\substack{p,q,m,n \\ p^m > q^n}} \frac{p^m}{p^m - q^n} p^{-ma} q^{-nb} &< \sum_{q,n} \sum_{l=1}^{\infty} \frac{l + q^n}{l} (l + q^n)^{-a} q^{-nb} \\
&= \sum_{q,n} \left(\sum_{l=1}^{\infty} \frac{1}{l} (q^n + l)^{-(a-1)} \right) q^{-nb} \\
&< \sum_{q,n} 2^{-(a-1)} \frac{a+1}{a-1} q^{-\frac{n}{2}(a-1)} q^{-nb} \\
&= 2^{-(a-1)} \frac{a+1}{a-1} \sum_{q,n} q^{-(\frac{a-1}{2}+b)n} \\
&< \infty \qquad (9.3)
\end{aligned}$$

if $\frac{a-1}{2} + b > 1$ and $a > 1$, which is equivalently $a + 2b > 3$ and $a > 1$. We obtain (9.1) by putting $a = \sigma - \alpha - \frac{1}{2}$ and $b = \alpha + \frac{1}{2}$.

Next for proving (9.2) we similarly have by putting $l = q^n - p^m \geq 1$

$$\begin{aligned}
\sum_{\substack{p,q,m,n \\ p^m < q^n}} \frac{q^n}{q^n - p^m} p^{-ma} q^{-nb} &< \sum_{p,m} \sum_{l=1}^{\infty} \frac{l + p^m}{l} (l + p^m)^{-b} p^{-ma} \\
&= \sum_{p,m} \left(\sum_{l=1}^{\infty} \frac{1}{l} (p^m + l)^{-(b-1)} \right) p^{-ma} \\
&\leq \sum_{p,m} 2^{-(b-1)} \frac{b+1}{b-1} p^{-(\frac{b-1}{2}+a)m} \\
&< \infty
\end{aligned}$$

if $\frac{b-1}{2} + a > 1$ and $b > 1$, which is equivalently $2a + b > 3$ and $b > 1$. We obtain (9.2) by putting $a = \sigma - \alpha - \frac{1}{2}$ and $b = \alpha + \frac{1}{2}$. This completes the proof of (1).

By considering $\frac{\partial}{\partial b}$(9.3), we similarly have (2). Hence Lemma 9.1. ∎

Now we calculate the multiple Euler product expression. From

$$\hat{\zeta}(s, \mathbf{Z}) = \zeta(s, \mathbf{Z}) \Gamma_{\mathbf{R}}(s)$$

we have

$$\hat{\zeta}(s, \mathbf{Z}) \otimes \hat{\zeta}(s, \mathbf{Z}) = (\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z})) (\zeta(s, \mathbf{Z}) \otimes \Gamma_{\mathbf{R}}(s))^2 (\Gamma_{\mathbf{R}}(s) \otimes \Gamma_{\mathbf{R}}(s))$$

with

$$\begin{aligned}
\Gamma_{\mathbf{R}}(s) \otimes \Gamma_{\mathbf{R}}(s) &\cong \prod_{m,n\geq 0} (s+2m+2n) \\
&\cong \prod_{m,n\geq 0} (\frac{s}{2}+m+n) \\
&\cong \Gamma_2\left(\frac{s}{2}\right)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\zeta(s,\mathbf{Z}) \otimes \Gamma_{\mathbf{R}}(s) &\cong \left(\prod_{\hat{\zeta}(\rho)=0} (s-\rho) \prod_{n=1}^{\infty}(s+2n) \times (s-1)^{-1}\right) \otimes \left(\prod_{m\geq 0}(s+2m)\right)^{-1} \\
&\cong \left(\prod_{\hat{\zeta}(\rho)=0} (s-\rho) \otimes \prod_{m\geq 0}(s+2m)^{-1}\right) \left(\prod_{n=1}^{\infty}(s+2n) \otimes \prod_{m\geq 0}(s+2m)^{-1}\right) \\
&\quad \left((s-1)^{-1} \otimes \prod_{m\geq 0}(s+2m)^{-1}\right) \\
&\cong \left(\prod_{\substack{\rho,m \\ \mathrm{Im}(\rho)>0}} (s-(\rho-2m))\right)^{-1} \left(\prod_{\substack{n\geq 1 \\ m\geq 0}}(s+2(n+m))\right)^{-1} \prod_{m\geq 0}(s+2m-1) \\
&\cong \prod_{m=0}^{\infty} \zeta_+(s+2m)^{-1}\Gamma_2\left(\frac{s+2}{2}\right)\Gamma_1\left(\frac{s-1}{2}\right)^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\zeta(s,\mathbf{Z}) \otimes \zeta(s,\mathbf{Z}) &= \left(\hat{\zeta}(s,\mathbf{Z}) \otimes \hat{\zeta}(s,\mathbf{Z})\right)(\zeta(s,\mathbf{Z}) \otimes \Gamma_{\mathbf{R}}(s))^{-2}(\Gamma_{\mathbf{R}}(s) \otimes \Gamma_{\mathbf{R}}(s))^{-1} \\
&\cong \left(\hat{\zeta}(s,\mathbf{Z}) \otimes \hat{\zeta}(s,\mathbf{Z})\right)\left(\prod_{m=0}^{\infty} \zeta_+(s+2m)^2\right)\Gamma_2\left(\frac{s+2}{2}\right)^{-2}\Gamma_1\left(\frac{s-1}{2}\right)^{2}\Gamma_2\left(\frac{s}{2}\right) \\
&\cong \left(\hat{\zeta}(s,\mathbf{Z}) \otimes \hat{\zeta}(s,\mathbf{Z})\right)\left(\prod_{m=0}^{\infty} \zeta_+(s+2m)^2\right)\Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2(s-1)^{-2}
\end{aligned}$$

since

$$\begin{aligned}
\Gamma_2\left(\frac{s+2}{2}\right)^{-2}\Gamma_1\left(\frac{s-1}{2}\right)^{2}\Gamma_2\left(\frac{s}{2}\right) &= \Gamma_2\left(\frac{s}{2}+1\right)^{-2}\Gamma_2\left(\frac{s}{2}\right)\Gamma_1\left(\frac{s-1}{2}\right)^{2} \\
&= \left(\Gamma_2\left(\frac{s}{2}\right)\Gamma_1\left(\frac{s}{2}\right)^{-1}\right)^{-2}\Gamma_2\left(\frac{s}{2}\right)\Gamma_1\left(\frac{s-1}{2}\right)^{2} \\
&= \Gamma_2\left(\frac{s}{2}\right)^{-1}\left(\Gamma_1\left(\frac{s}{2}\right)\Gamma_1\left(\frac{s-1}{2}\right)\right)^{2} \\
&\cong \Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s-1)^2 \\
&= \Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2(s-1)^{-2},
\end{aligned}$$

where we used the relation

$$\Gamma_r(x+1) = \Gamma_r(x)\Gamma_{r-1}(x)^{-1}$$

(see [KK2]). Note that

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}$$

and

$$\Gamma_0(x) = x^{-1}.$$

We used also the duplication formula

$$\Gamma_1\left(\frac{x}{2}\right)\Gamma_1\left(\frac{x+1}{2}\right) \cong \Gamma_1(x).$$

On the other hand, from §4 we recall the formula (4.2) (after the translation $s \to s-1$)

$$\hat{\zeta}(s,\mathbf{Z}) \otimes \hat{\zeta}(s,\mathbf{Z}) \cong \frac{L(s-1)}{L(1-s)}\zeta_+(s)^{-2}\zeta_+(s-1)^{-2}(s-2)(s-1)^2 s$$

and

$$\frac{L(s-1)}{L(1-s)} \cong \prod_{p,q}\zeta_{p,q}^{\alpha}(s) \times \zeta_0^{\alpha}(s).$$

Hence

$$\begin{aligned}
\zeta(s,\mathbf{Z}) \otimes \zeta(s,\mathbf{Z}) &\cong \left(\prod_{p,q}\zeta_{p,q}^{\alpha}(s)\right)\zeta_0^{\alpha}(s)\zeta_+(s-1)^{-2}\zeta_+(s)^{-2}(s-2)(s-1)^2 s \\
&\quad \times \left(\prod_{m=0}^{\infty}\zeta_+(s+2m)^2\right)\Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2(s-1)^{-2} \\
&= \left(\prod_{p,q}\zeta_{p,q}^{\alpha}(s)\right)\zeta_0^{\alpha}(s)\left(\frac{\prod_{m=1}^{\infty}\zeta_+(s+2m)}{\zeta_+(s-1)}\right)^2\Gamma_2\left(\frac{s}{2}\right)^{-1}\Gamma_1(s)^2(s-2)s.
\end{aligned}$$

Lastly, the functional equation $s \to 2-s$ is obtained from the invariance

$$\frac{L(s-1)}{L(1-s)} = \left(\frac{L((2-s)-1)}{L(1-(2-s))}\right)^{-1}.$$

Remark 9.3 From the applicational viewpoint (see Problem (2) in §10 below) it is quite interesting to see the convergence region of

$$\zeta_0(s,\mathbf{Z}) \otimes \zeta_0(s,\mathbf{Z}) = \left(\prod_{p,q} \zeta_{p,q}^{\alpha}(s)\right) \zeta_0^{\alpha}(s) \left(\prod_{m=1}^{\infty} \zeta_+(s+2m)\right)^2 \Gamma_2\left(\frac{s}{2}\right)^{-1} \Gamma_1(s)^2 \Gamma_1\left(\frac{s+1}{2}\right)^{-2} s.$$

Does this converge in $\mathrm{Re}(s) > \alpha + 1$? Here $\zeta_0(s,\mathbf{Z}) = (s-1)\zeta(s,\mathbf{Z})$.

10 Problems

We list some of the remaining problems.

(1) Higher tensor products:

We studied above $Z_1(s) \otimes Z_2(s)$ only. It is quite interesting to investigate $Z_1(s) \otimes \cdots \otimes Z_r(s)$ for $r \geq 3$: for example, $\zeta(s,\mathbf{Z}) \otimes \zeta(s,\mathbf{Z}) \otimes \zeta(s,\mathbf{Z})$. Our method is extendable to such cases but the needed calculations become cumbersome. Akatsuka [A] studied $\zeta(s,\mathbf{F}_{p_1}) \otimes \zeta(s,\mathbf{F}_{p_2}) \otimes \zeta(s,\mathbf{F}_{p_3})$ for distinct primes p_1, p_2 and p_3 by our method, and he obtained the following Euler product (or the "Euler factor"):

Theorem (Akatsuka [A]) *Let $p,\ q,\ r$ be distinct primes. In* $\mathrm{Re}(s)>0$ *we have*

$$\begin{aligned}
&\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)\otimes\zeta(s,\mathbf{F}_r)\\
&= e^{Q(s)}(1-p^{-s})^{-\frac{1}{4}}(1-q^{-s})^{-\frac{1}{4}}(1-r^{-s})^{-\frac{1}{4}}\\
&\quad \exp\left(-\frac{1}{4}\sum_{n_1=1}^{\infty}\frac{\cot\left(\pi n_1\frac{\log p}{\log q}\right)\cot\left(\pi n_1\frac{\log p}{\log r}\right)}{n_1}p^{-n_1 s}\right.\\
&\qquad -\frac{1}{4}\sum_{n_2=1}^{\infty}\frac{\cot\left(\pi n_2\frac{\log q}{\log p}\right)\cot\left(\pi n_2\frac{\log q}{\log r}\right)}{n_2}q^{-n_2 s}\\
&\qquad -\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{\cot\left(\pi n_3\frac{\log r}{\log p}\right)\cot\left(\pi n_3\frac{\log r}{\log q}\right)}{n_3}r^{-n_3 s}\\
&\qquad +\frac{i}{4}\sum_{n_1=1}^{\infty}\frac{\cot\left(\pi n_1\frac{\log p}{\log q}\right)+\cot\left(\pi n_1\frac{\log p}{\log r}\right)}{n_1}p^{-n_1 s}\\
&\qquad +\frac{i}{4}\sum_{n_2=1}^{\infty}\frac{\cot\left(\pi n_2\frac{\log q}{\log p}\right)+\cot\left(\pi n_2\frac{\log q}{\log r}\right)}{n_2}q^{-n_2 s}\\
&\qquad \left.+\frac{i}{4}\sum_{n_3=1}^{\infty}\frac{\cot\left(\pi n_3\frac{\log r}{\log p}\right)+\cot\left(\pi n_3\frac{\log r}{\log q}\right)}{n_3}r^{-n_3 s}\right),
\end{aligned}$$

where $Q(s)$ is a polynomial of degree at most 3.

Moreover, Akatsuka calculated the degenerate cases (p,p,p) and (p,p,q). After Akatsuka's paper [A], the general case $\zeta(s,\mathbf{F}_{p_1})\otimes\cdots\otimes\zeta(s,\mathbf{F}_{p_r})$ for distinct primes $p_1,...,p_r$ is treated in [KW1] by a different method.

(2) Obtaining results on essential zeros:

If we can prove that $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ (or $\zeta_0(s,\mathbf{Z})\otimes\zeta_0(s,\mathbf{Z})$ in Remark 9.3) has no essential zeros and poles in $\mathrm{Re}(s)>\frac{3}{2}$, then we have $\frac{1}{4}\le\mathrm{Re}(\rho)\le\frac{3}{4}$ for all essential zeros ρ of $\zeta(s,\mathbf{Z})$. More generally if we know that there are no essential zeros and poles of $\zeta(s,\mathbf{Z})^{\otimes r}$ for $r\ge 2$ in $\mathrm{Re}(s)>\frac{r+1}{2}$, then we obtain $\frac{r-1}{2r}\le\mathrm{Re}(\rho)\le\frac{r+1}{2r}$ for all essential zeros ρ of $\zeta(s,\mathbf{Z})$. Thus letting $r\to\infty$ we would have $\mathrm{Re}(\rho)=\frac{1}{2}$. (See Grothendieck [G] and Deligne [D] for the case of a scheme over $\mathbf{F}_p$.)

We may expect that our Euler product for $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ (or $\zeta_0(s,\mathbf{Z})\otimes\zeta_0(s,\mathbf{Z})$) should imply such a needed result for $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ (or $\zeta_0(s,\mathbf{Z})\otimes\zeta_0(s,\mathbf{Z})$). But at present we do not have results in this direction. Does this consideration have some relations to the "absolute scheme" $\mathrm{Spec}(\mathbf{Z}\otimes_{\mathbf{F}_1}\mathbf{Z})$ as in [KOW] and [Dei]?

(3) Theory of multiple Euler products:

Is there a theory of multiple Euler products implying

$$\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z}) = \prod_{p,q} \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$$

perfectly with functorial relations? We see some discrepancy from our present results.

Note (May 2006): Recetnly H. Akatsuka (Tokyo Institute of Technology) calculated the case "$\alpha = \frac{1}{2}$". His method gives the double Euler product in a direct way; see H. Akatsuka "The double Riemann zeta function" (in preparation).

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