Multiple Eisenstein Series and Multiple Cotangent Functions

Shin-ya Koyama and Nobushige Kurokawa

Abstract. We construct the multiple Eisenstein series and we show a relation to the multiple cotangent function. We calculate a limit value of the multiple Eisenstein series.

Key words: multiple Eisenstein series, multiple cotangent function, multiple polylogarithm

AMS Classification 11M06

1 Introduction

For integers $r \geq 1$ and $k \geq r+2$, we define the multiple Eisenstein series as

$$F_k(\tau_1, \cdots, \tau_r) = \sum_{n=1}^{\infty} \frac{n^{k-1} q_1^n \cdots q_r^n}{(1-q_1^n)\cdots(1-q_r^n)}$$

with $q_j = e^{2\pi i \tau_j}$ for $\mathrm{Im}(\tau_j) > 0$. We remark that the series converges for $(\tau_1, \cdots, \tau_r) \in (\mathbf{C} - \mathbf{R}_{\leq 0})^r$ if $\mathrm{Im}(\tau_j) > 0$ for at least one j. This function is considered to be the multiple q (quantum) polylogarithm since

$$F_k(\tau_1, \cdots, \tau_r) = \frac{1}{(1-q_1)\cdots(1-q_r)} \sum_{n=1}^{\infty} \frac{q_1^n \cdots q_r^n}{n^{1-k}[n]_{q_1} \cdots [n]_{q_r}},$$

where

$$[x]_q = \frac{1-q^x}{1-q}.$$

We recall the one-variable case

$$F_k(\tau) = \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with

$$\sigma_{k-1}(n)=\sum_{d\,|\,n} d^{k-1}$$

and

$$q=e^{2\pi i\tau}$$

occurs frequently in the form

$$E_k(\tau)=\frac{\zeta(1-k)}{2}+\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

with the constant term $\zeta(1-k)/2$.

In a previous paper [K2] we proved that

$$\lim_{\substack{\tau\to 1\\ \mathrm{Im}(\tau)>0}}\left(E_k\left(-\frac{1}{\tau}\right)-\tau^k E_k(\tau)\right)=\frac{(-1)^k B_{k-1}}{2\pi i}$$

for all positive integers k. This is checked for even integers k by the modularity of $E_k(\tau)$ for $SL_2(\mathbf{Z})$, but the above limit value seems to be curious especially for odd integers k. We remark that the above result is written also as

$$\lim_{\substack{\tau\to 1\\ \mathrm{Im}(\tau)>0}}\left(F_k\left(-\frac{1}{\tau}\right)-\tau^k F_k(\tau)\right)=\frac{(-1)^k B_{k-1}}{2\pi i}.$$

In this paper we generalize this result to the case of several variables.

To describe our results we introduce multiple sine functions and multiple cotangent functions. The multiple sine function $S_r(x,(\omega_1,\cdots,\omega_r))$ is constructed as

$$\begin{aligned}&S_r(x,(\omega_1,\cdots,\omega_r))\\ &=\coprod_{n_1,\cdots,n_r\geq 0}(n_1\omega_1+\cdots+n_r\omega_r+x)\left(\coprod_{m_1,\cdots,m_r\geq 1}(m_1\omega_1+\cdots+m_r\omega_r-x)\right)^{(-1)^{r-1}},\end{aligned}$$

where $\coprod$ denotes the regularized product of Deninger [D]:

$$\coprod_{\lambda}\lambda=\exp\left(-\frac{d}{ds}\sum_{\lambda}\lambda^{-s}\Big|_{s=0}\right).$$

We refer to [K1] [KK] for detailes of the theory of multiple sine functions; see the survey of Manin [M]. The multiple cotangent function $\mathrm{Cot}_r(x,(\omega_1,\cdots,\omega_r))$ is defined as the logarithmic derivative of the multiple sine function:

$$\mathrm{Cot}_r(x,(\omega_1,\cdots,\omega_r))=\frac{S_r'(x,(\omega_1,\cdots,\omega_r))}{S_r(x,(\omega_1,\cdots,\omega_r))}.$$

Our first result relates $F_k(\tau_1,\cdots,\tau_r)$ to $\mathrm{Cot}_{r+1}^{(k-1)}(\tau_1+\cdots+\tau_r,(\tau_1,\cdots,\tau_r,1))$.

Theorem 1 *Let $k \geq r+2$ and $0 < \arg(\tau_1) < \cdots < \arg(\tau_r) < \pi$. Then*

$$\begin{aligned}
&\mathrm{Cot}_{r+1}^{(k-1)}(\tau_1 + \cdots + \tau_r, (\tau_1, \cdots, \tau_r, 1)) \\
&= \ -(2\pi i)^k \left(F_k(\tau_1, \cdots, \tau_r) - \frac{1}{\tau_1^k} F_k\left(-\frac{1}{\tau_1}, \frac{\tau_2}{\tau_1}, \cdots, \frac{\tau_r}{\tau_1}\right) - \frac{1}{\tau_2^k} F_k\left(\frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2}, \frac{\tau_3}{\tau_2}, \cdots, \frac{\tau_r}{\tau_2}\right) \right. \\
&\quad \left. - \cdots - \frac{1}{\tau_r^k} F_k\left(\frac{\tau_1}{\tau_r}, \cdots, \frac{\tau_{r-1}}{\tau_r}, -\frac{1}{\tau_r}\right)\right).
\end{aligned}$$

Examples

(1)

$$\mathrm{Cot}_2^{(k-1)}(\tau, (\tau, 1)) = -(2\pi i)^k \left(F_k(\tau) - \frac{1}{\tau^k} F_k\left(-\frac{1}{\tau}\right)\right).$$

(2)

$$\mathrm{Cot}_3^{(k-1)}(\tau + \sigma, (\tau, \sigma, 1)) = -(2\pi i)^k \left(F_k(\tau, \sigma) - \frac{1}{\tau^k} F_k\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) - \frac{1}{\sigma^k} F_k\left(\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right)\right).$$

Theorem 2 *Let $k \geq r+2$. Then*

$$\begin{aligned}
&\lim_{\tau_1, \cdots, \tau_r \to 1} \left(F_k(\tau_1, \cdots, \tau_r) - \frac{1}{\tau_1^k} F_k\left(-\frac{1}{\tau_1}, \frac{\tau_2}{\tau_1}, \cdots, \frac{\tau_r}{\tau_1}\right) - \cdots - \frac{1}{\tau_r^k} F_k\left(\frac{\tau_1}{\tau_r}, \cdots, \frac{\tau_{r-1}}{\tau_r}, -\frac{1}{\tau_r}\right)\right) \\
&= \ \frac{(-1)^{r-1}(k-1)!}{r!(2\pi i)^k} \sum_{m=0}^{[\frac{k-r}{2}]} \frac{(-1)^m \pi^{2m} a(r, k-2m)}{(2m)!} B_{2m},
\end{aligned}$$

where $a(r,j) \in \mathbf{Z}$ is a Stirling like number defined by

$$x(x+1)\cdots(x+r-1) = \sum_{j=1}^{n} a(r,j) x^j.$$

Examples

(1)

$$\lim_{\substack{\tau \to 1 \\ \mathrm{Im}(\tau) > 0}} \left(F_k(\tau) - \frac{1}{\tau^k} F_k\left(-\frac{1}{\tau}\right)\right) = \frac{(-1)^{k-1} B_{k-1}}{2\pi i}.$$

(2)

$$\lim_{\substack{\tau,\sigma\to 1\\ 0<\arg(\tau)<\arg(\sigma)<\pi}} \left(F_k(\tau,\sigma) - \frac{1}{\tau^k} F_k\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) - \frac{1}{\sigma^k} F_k\left(\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right)\right)$$

$$= \begin{cases} -\dfrac{B_{k-1}}{4\pi i} & \text{if } k \geq 1 \text{ is odd,} \\[2ex] \dfrac{(k-1)B_{k-2}}{8\pi^2} & \text{if } k \geq 2 \text{ is even.} \end{cases}$$

Our functions can be investigated from the viewpoint of the "Stirling modular form" which generalizes a notion used by Barnes [B]. Instead of going into the detailed theory, here we only indicate that "Stirling modular forms" mean some suitable functions of "semi-lattices" similar to the situation of usual modular forms which are functions of "lattices". We notice one example which shows that $F_k(\tau_1, \cdots, \tau_r)$ is a function of the "semi-lattice" $\mathbf{Z}_{\geq 1}\tau_1 + \cdots + \mathbf{Z}_{\geq 1}\tau_r + \mathbf{Z} \cdot 1$.

Theorem 3 *Let $k \geq r+2$ and $\mathrm{Im}(\tau_1), \cdots, \mathrm{Im}(\tau_r) > 0$. Then*

$$F_k(\tau_1, \cdots, \tau_r) = \frac{(-1)^k (k-1)!}{(2\pi i)^k} \sum_{m_1, \cdots, m_r \geq 1} \sum_{n=-\infty}^{\infty} \frac{1}{(m_1\tau_1 + \cdots + m_r\tau_r + n)^k}.$$

2 Proof of Theorem 1

As proved in [KW] we have

$$\log S_r(x, (\omega_1, \cdots, \omega_r)) = -\sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi i n x}{\omega_l}}}{n \prod_{j\neq l} \left(1 - e^{\frac{2\pi i n \omega_j}{\omega_l}}\right)} + Q(x),$$

where $Q(x)$ is a polynomial of x with $\deg Q \leq r$. Hence we have

$$\mathrm{Cot}_r(x, (\omega_1, \cdots, \omega_r)) = -2\pi i \sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi i n x}{\omega_l}}}{\omega_l \prod_{j\neq l} \left(1 - e^{\frac{2\pi i n \omega_j}{\omega_l}}\right)} + Q'(x)$$

and

$$\mathrm{Cot}_r^{(k-1)}(x, (\omega_1, \cdots, \omega_r)) = -(2\pi i)^k \sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{n^{k-1} e^{\frac{2\pi i n x}{\omega_l}}}{\omega_l^k \prod_{j\neq l} \left(1 - e^{\frac{2\pi i n \omega_j}{\omega_l}}\right)} + Q^{(k)}(x)$$

for $k \geq 1$. In particular

$$\mathrm{Cot}_r^{(k-1)}(x,(\omega_1,\cdots,\omega_r)) = -(2\pi i)^k \sum_{l=1}^{r}\sum_{n=1}^{\infty} \frac{n^{k-1}e^{\frac{2\pi i n x}{\omega_l}}}{\omega_l^k \prod_{j\neq l}\left(1-e^{\frac{2\pi i n \omega_j}{\omega_l}}\right)}$$

for $k \geq r+1$. Thus, changing r to $r+1$ and taking $(\omega_1,\cdots,\omega_{r+1}) = (\tau_1,\cdots,\tau_r,1)$ with $x = \tau_1+\cdots+\tau_r$, we obtain Theorem 1. ∎

3 Proof of Theorem 2

Theorem 1 shows that the limit value is given by

$$-\frac{1}{(2\pi i)^k}\mathrm{Cot}_{r+1}^{(k-1)}(r,(1,\cdots,1)).$$

We calculate it by using the formula

$$\mathrm{Cot}_{r+1}(x,(1,\cdots,1)) = (-1)^r\binom{x-1}{r}\pi\cot(\pi x)$$

proved in [KK]. We look at

$$\mathrm{Cot}_{r+1}(r+x,(1,\cdots,1)) = (-1)^r\binom{x+r-1}{r}\pi\cot(\pi x)$$

and its Taylor expansion around $x=0$. Since

$$\binom{x+r-1}{r} = \frac{x(x+1)\cdots(x+r-1)}{r!} = \frac{1}{r!}\sum_{j=1}^{r}a(r,j)x^j$$

and

$$\cot(\pi x) = \sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m-1}B_{2m}}{(2m)!}x^{2m-1}$$

we have

$$\begin{aligned}\mathrm{Cot}_{r+1}^{(k-1)}(r,(1,\cdots,1)) &= (k-1)!\frac{(-1)^r\pi}{r!}\sum_{j=1}^{r}\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m-1}B_{2m}a(r,j)}{(2m)!}\\ &= (k-1)!\frac{(-1)^r}{r!}\sum_{m=0}^{[\frac{k-r}{2}]}\frac{(-1)^m\pi^{2m}a(r,k-2m)}{(2m)!}B_{2m}.\end{aligned}$$

Thus we have Theorem 2. ∎

We notice that Examples are easily checked by Theorem 1.

4 Proof of Theorem 3

The well-known Lipschitz formula says that

$$\sum_{n=-\infty}^{\infty} (\tau+n)^{-k} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n$$

for $\mathrm{Im}(\tau) > 0$ with $q = e^{2\pi i\tau}$. Especially, for $\tau = m_1\tau_1 + \cdots + m_r\tau_r$ we have

$$\sum_{n=-\infty}^{\infty} (m_1\tau_1 + \cdots + m_r\tau_r + n)^{-k} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} (q_1^{m_1} \cdots q_r^{m_r})^n$$

with $q_j = e^{2\pi i\tau_j}$. Hence we obtain

$$\sum_{m_1,\cdots,m_r \geq 1} \sum_{n=-\infty}^{\infty} (m_1\tau_1 + \cdots + m_r\tau_r + n)^{-k} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \frac{n^{k-1} q_1^n \cdots q_r^n}{(1-q_1^n)\cdots(1-q_r^n)}.$$

This gives Theorem 3. ∎

References

[B] Barnes, E.W.: On the theory of the multiple gamma function. Trans. Cambridge Philos. Soc. **19** (1904) 374-425.

[D] Deninger, C.: Local L-factors of motives and regularized determinant. Invent. Math. **107** (1992) 135-150.

[K1] Kurokawa, N.: Gamma factors and Plancherel measures. Proc. Japan Acad. **68A** (1992) 256-260.

[K2] Kurokawa, N.: Limit values of Eisenstein series and multiple cotangent functions (preprint 2006).

[KK] Kurokawa, N., and Koyama, S.: Multiple sine functions. Forum Math. **15** (2003) 839-876.

[KW] Kurokawa, N., and Wakayama, M.: Absolute tensor products. Int. Math. Res. Not. **2004-5** (2004) 249-260.

[M] Manin, Yu. I.: Lectures on zeta functions and motives (according to Deninger and Kurokawa). Asterisque, **228** (1995) 121-163.

Department of Mathematics, Tokyo Institute of Technology
Oo-okayama, Meguro-ku, Tokyo 152-8551, Japan
kurokawa@math.titech.ac.jp