Euler's Integrals and Multiple Sine Functions

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Running title. Euler's Integrals

Abstract. We show that Euler's famous integrals whose integrands contain the logarithm of the sine function are expressed via multiple sine functions.

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1 Introduction

Euler studied the definite integrals $\int_0^{\frac{\pi}{2}} x^n \log(\sin x)dx$ for $n = 0$ and 1. In [E1] (1769), he proved the famous result

$$\int_0^{\frac{\pi}{2}} \log(\sin x)dx = -\frac{\pi}{2}\log 2, \tag{1.1}$$

which is frequently explained as an example of tricky integrals in analysis courses. A bit later, Euler [E2] (1772) stated that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\log(\sin x)dx &= \frac{1}{2}\left(\sum_{\substack{n=1\\ n:\text{odd}}}^{\infty}\frac{1}{n^3} - \frac{\pi^2}{4}\log 2\right)\\ &= \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\log 2. \end{aligned}\tag{1.2}$$

Euler proved (1.1) by using

$$\log(\sin x) = -\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n} - \log 2 \tag{1.3}$$

for $0 < x < \pi$. The actual integration is obvious since

$$\int_0^{\frac{\pi}{2}} \cos(2nx)dx = 0$$

for $n = 1, 2, 3, \cdots$. Hence, the original proof of Euler is not tricky contrary to the usual explanation. In the case of (1.2), Euler primary wanted to calculate the value $\zeta(3)$. He started from the divergent series expression

$$\begin{aligned}\zeta(3) &= -4\pi^2\zeta'(-2) \\ &= 4\pi^2\sum_{n=1}^{\infty} n^2 \log n\end{aligned}$$

and by investigating it he reached (1.2). Thus his arguments are difficult to follow and literally invalid. Moreover Euler did not prove the functional equation $\zeta(3) = -4\pi^2\zeta'(-2)$ conjectured by himself in [E3]. We notice that when we use (1.3) we can give a secure calculation for (1.2):

$$\int_0^{\frac{\pi}{2}} x \log(\sin x) dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} x \cos(2nx) dx - \frac{\pi^2}{8} \log 2$$

with integration by parts

$$\int_0^{\frac{\pi}{2}} x \cos(2nx) dx = \begin{cases} -\dfrac{1}{2n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

It might be remarkable that Euler missed this way.

In this paper we investigate definite integrals

$$\int_0^x \theta^{r-2} \log(\sin\theta) d\theta$$

for $r = 2, 3, 4, \cdots$ containing the Euler's case $x = \pi/2$ from the point of view of multiple sine functions. Let

$$\begin{aligned}\mathscr{S}_r(x) &= e^{\frac{x^{r-1}}{r-1}} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} P_r\left(\frac{x}{n}\right)^{n^{r-1}} \\ &= e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left(P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}\end{aligned}$$

be the multiple sine function studied in [K1, K2, KK1, KK2, KOW, KW], where

$$P_r(u) = (1-u) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

For example

$$\mathscr{S}_2(x) = e^x \prod_{n=1}^{\infty} \left(\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^n e^{2x} \right),$$

$$\mathscr{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left(\left(1 - \frac{x^2}{n^2} \right)^{n^2} e^{x^2} \right)$$

and

$$\mathscr{S}_4(x) = e^{\frac{x^3}{3}} \prod_{n=1}^{\infty} \left(\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}} \right)^{n^3} e^{2n^2x+\frac{2}{3}x^3} \right).$$

Then we show the following result.

Theorem 1 *For $0 \le x < \pi$ and for $r = 2, 3, 4, ...,$ we have*

$$\int_0^x \theta^{r-2} \log(\sin\theta) d\theta = \frac{x^{r-1}}{r-1} \log(\sin x) - \frac{\pi^{r-1}}{r-1} \log \mathscr{S}_r \left(\frac{x}{\pi} \right).$$

In particular we have:

Theorem 2 *For $r = 2, 3, 4, ...,$*

$$\int_0^{\frac{\pi}{2}} \theta^{r-2} \log(\sin\theta) d\theta = -\frac{\pi^{r-1}}{r-1} \log \mathscr{S}_r \left(\frac{1}{2} \right).$$

Examples.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \log(\sin\theta) d\theta &= -\pi \log \mathscr{S}_2 \left(\frac{1}{2} \right) \\ &= -\pi \log \left(e^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(\left(\frac{2n-1}{2n+1} \right)^n e \right) \right), \end{aligned} \tag{1.4}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \theta \log(\sin\theta) d\theta &= -\frac{\pi^2}{2} \log \mathscr{S}_3 \left(\frac{1}{2} \right) \\ &= -\frac{\pi^2}{2} \log \left(e^{\frac{1}{8}} \prod_{n=1}^{\infty} \left(\left(1 - \frac{1}{4n^2} \right)^{n^2} e^{\frac{1}{4}} \right) \right), \end{aligned} \tag{1.5}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \theta^2 \log(\sin\theta) d\theta &= -\frac{\pi^3}{3} \log \mathscr{S}_4 \left(\frac{1}{2} \right) \\ &= -\frac{\pi^3}{3} \log \left(e^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(\left(\frac{2n-1}{2n+1} \right)^{n^3} \exp\left(n^2 + \frac{1}{12} \right) \right) \right). \end{aligned} \tag{1.6}$$

We notice that we have $\mathscr{S}_2\left(\frac{1}{2}\right) = \sqrt{2}$ and $\mathscr{S}_3\left(\frac{1}{2}\right) = 2^{\frac{1}{4}}\exp\left(-\frac{7\zeta(3)}{8\pi^2}\right)$ from Euler's results (1.1) and (1.2) combined with (1.4) and (1.5). We demonstrate a calculation of the special value $\mathscr{S}_4\left(\frac{1}{2}\right)$ from the product expression directly as follows.

Theorem 3

$$\mathscr{S}_4\left(\frac{1}{2}\right) = 2^{\frac{1}{8}}\exp\left(-\frac{9\zeta(3)}{16\pi^2}\right)$$

and

$$\int_0^{\frac{\pi}{2}} \theta^2 \log(\sin\theta)d\theta = \frac{3\pi}{16}\zeta(3) - \frac{\pi^3}{24}\log 2.$$

In our calculation a generalization of the Stirling formula is crucial.

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2 Multiple Sine Functions

To make this paper self-contained we prove some basic properties of multiple sine functions. For general background we refer to [KK1, KK2, KOW, M].

Proposition 1 *For $r = 2, 3, 4, ...,$ $\mathscr{S}_r(x)$ is a meromorphic function in $x \in \mathbf{C}$ and it satisfies*

$$\frac{\mathscr{S}_r'(x)}{\mathscr{S}_r(x)} = \pi x^{r-1}\cot(\pi x).$$

Proof. The fact that $\mathscr{S}_r(x)$ is meromorphic function in $x \in \mathbf{C}$ (and its order as a meromorphic function being r) is seen from the product expression defining $\mathscr{S}_r(x)$. Let us calculate the logarithmic derivative. From

$$\mathscr{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left(P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}$$

we have

$$\begin{aligned}
\log \mathscr{S}_r(x) &= \frac{x^{r-1}}{r-1} + \sum_{n=1}^{\infty} n^{r-1}\left(\log P_r\left(\frac{x}{n}\right) + (-1)^{r-1}\log P_r\left(-\frac{x}{n}\right)\right) \\
&= \frac{x^{r-1}}{r-1} + \sum_{n=1}^{\infty} n^{r-1}\Bigg(\log\left(1-\frac{x}{n}\right) + (-1)^{r-1}\log\left(1+\frac{x}{n}\right) \\
&\quad + \left(\frac{x}{n} + \frac{1}{2}\left(\frac{x}{n}\right)^2 + \cdots + \frac{1}{r}\left(\frac{x}{n}\right)^r\right) \\
&\quad + (-1)^{r-1}\left(\left(\frac{-x}{n}\right) + \frac{1}{2}\left(\frac{-x}{n}\right)^2 + \cdots + \frac{1}{r}\left(\frac{-x}{n}\right)^r\right)\Bigg).
\end{aligned}$$

Hence

$$\frac{\mathscr{S}_r'(x)}{\mathscr{S}_r(x)} = x^{r-2} + \sum_{n=1}^{\infty} n^{r-1}\left(\frac{1}{x-n} + \frac{(-1)^{r-1}}{x+n} + \left(\frac{1}{n} + \frac{x}{n^2} + \cdots + \frac{x^{r-1}}{n^r}\right)\right.$$
$$\left. +(-1)^{r-1}\left(-\frac{1}{n} + \frac{x}{n^2} + \cdots + (-1)^r \frac{x^{r-1}}{n^r}\right)\right).$$

Here

$$\frac{1}{n} + \frac{x}{n^2} + \cdots + \frac{x^{r-1}}{n^r} = \frac{(\frac{x}{n})^r - 1}{x-n}$$

and

$$-\frac{1}{n} + \frac{x}{n^2} + \cdots + (-1)^r \frac{x^{r-1}}{n^r} = \frac{(-1)^r(\frac{x}{n})^r - 1}{x+n}.$$

Thus

$$\begin{aligned}
\frac{\mathscr{S}_r'(x)}{\mathscr{S}_r(x)} &= x^{r-2} + \sum_{n=1}^{\infty} n^{r-1}\left(\frac{(\frac{x}{n})^r}{x-n} - \frac{(\frac{x}{n})^r}{x+n}\right) \\
&= x^{r-2} + \sum_{n=1}^{\infty} \frac{2x^r}{x^2-n^2} \\
&= \pi x^{r-1}\cot(\pi x),
\end{aligned}$$

where we used

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2-n^2}. \blacksquare$$

Proposition 2 *For* $0 \le x < 1$ *and for* $r = 2, 3, 4, \ldots,$

$$\log \mathscr{S}_r(x) = \int_0^x \pi t^{r-1}\cot(\pi t)dt.$$

Proof. Since $\mathscr{S}_r(0) = 1$, both sides are 0 at $x = 0$. Hence it is sufficient to remark that the differentiations of both sides are $\pi x^{r-1}\cot(\pi x)$ from Proposition 1. $\blacksquare$

3 Euler's Integrals

Using Proposition 2 we show Theorems 1 and 2.
Proof of Theorems 1 and 2: By integration by parts in

$$\log \mathscr{S}_r(x) = \int_0^x \pi t^{r-1}\cot(\pi t)dt$$

we have

$$\begin{aligned}
\log \mathscr{S}_r(x) &= \left[t^{r-1}\log(\sin \pi t)\right]_0^x - \int_0^x (r-1)t^{r-2}\log(\sin \pi t)dt \\
&= x^{r-1}\log(\sin \pi x) - (r-1)\int_0^x t^{r-2}\log(\sin \pi t)dt.
\end{aligned}$$

Hence changing the variable to $\theta = \pi t$ in the integral, we have

$$\log \mathscr{S}_r(x) = x^{r-1}\log(\sin \pi x) - \frac{r-1}{\pi^{r-1}}\int_0^{\pi x} \theta^{r-2}\log(\sin\theta)d\theta.$$

This gives Theorem 1. Then, letting $x = 1/2$ we have Theorem 2. ∎

Examples:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \log(\sin\theta)d\theta &= -\frac{\pi}{8}\log 2 - \pi\log\mathscr{S}_2\left(\frac{1}{4}\right), \\
\int_0^{\frac{\pi}{3}} \log(\sin\theta)d\theta &= \frac{\pi}{3}\log\frac{\sqrt{3}}{2} - \pi\log\mathscr{S}_2\left(\frac{1}{3}\right), \\
\int_0^{\frac{\pi}{4}} \theta\log(\sin\theta)d\theta &= -\frac{\pi^2}{64}\log 2 - \frac{\pi^2}{2}\log\mathscr{S}_3\left(\frac{1}{4}\right), \\
\int_0^{\frac{\pi}{3}} \theta\log(\sin\theta)d\theta &= \frac{\pi^2}{18}\log\frac{\sqrt{3}}{2} - \frac{\pi^2}{2}\log\mathscr{S}_3\left(\frac{1}{3}\right).
\end{aligned}$$

4 A calculation of the special value

Proof of Theorem 3: Since

$$\mathscr{S}_4\left(\frac{1}{2}\right) = e^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(\left(\frac{2n-1}{2n+1}\right)^{n^3}\exp\left(n^2+\frac{1}{12}\right)\right),$$

we put

$$\begin{aligned}
A_N &= e^{\frac{1}{24}}\prod_{n=1}^{N}\left(\left(\frac{2n-1}{2n+1}\right)^{n^3}\exp\left(n^2+\frac{1}{12}\right)\right) \\
&= \exp\left(\frac{1}{24}+(1^2+\cdots+N^2)+\frac{N}{12}\right)\times\prod_{n=1}^{N}(2n-1)^{n^3-(n-1)^3} \\
&\quad \times(2N+1)^{-N^3}
\end{aligned}$$

and show

$$\lim_{N\to\infty} A_N = 2^{\frac{1}{8}} \exp\left(\frac{9}{4}\zeta'(-2)\right) = 2^{\frac{1}{8}} \exp\left(-\frac{9\zeta(3)}{16\pi^2}\right).$$

Then the value of the integral follows from (1.6). We use the Stirling formula

$$N! = \prod_{n=1}^{N} n \sim \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N}$$

and its generalization

$$\prod_{n=1}^{N} n^{n^2} \sim \exp\left(-\zeta'(-2)\right) N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}} e^{-\frac{N^3}{9}+\frac{N}{12}},$$

which follow from the Euler-Maclaurin summation formula for $\zeta'(s)$:

$$\begin{aligned}\zeta'(s) = \lim_{N\to\infty} \Bigg(-\sum_{n=1}^{N} n^{-s}\log n + \frac{N^{1-s}\log N}{1-s} - \frac{N^{1-s}}{(1-s)^2} \\ +\frac{1}{2}N^{-s}\log N - \frac{s}{12}N^{-s-1}\log N + \frac{1}{12}N^{-s-1}\Bigg)\end{aligned}$$

valid in $\mathrm{Re}(s) > -3$. We refer to Hardy [H] Chap. XIII for the Euler-Maclaurin summation formula and its applications. Then, letting $s = 0$ and -2 we see

$$\begin{aligned}\log\sqrt{2\pi} = -\zeta'(0) &= \lim_{N\to\infty}\left(\sum_{n=1}^{N}\log n - \left(\left(N+\frac{1}{2}\right)\log N - N\right)\right) \\ &= \lim_{N\to\infty}\log\left(\frac{N!}{N^{N+\frac{1}{2}}e^{-N}}\right)\end{aligned}$$

and

$$\begin{aligned}-\zeta'(-2) &= \lim_{N\to\infty}\left(\sum_{n=1}^{N} n^2\log n - \left(\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\log N - \frac{N^3}{9} + \frac{N}{12}\right)\right) \\ &= \lim_{N\to\infty}\log\left(\frac{\prod_{n=1}^{N} n^{n^2}}{N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}} e^{-\frac{N^3}{9}+\frac{N}{12}}}\right).\end{aligned}$$

Using

$$\begin{aligned}
\prod_{n=1}^{N}(2n-1)^{n^3-(n-1)^3} &= \prod_{n=1}^{N}(2n-1)^{3n^2-3n+1} \\
&= \left(\prod_{n=1}^{N}(2n-1)^{(2n-1)^2}\right)^{\frac{3}{4}} \times \left(\prod_{n=1}^{N}(2n-1)\right)^{\frac{1}{4}} \\
&= \left(\frac{\prod_{n=1}^{2N} n^{n^2}}{\prod_{n=1}^{N}(2n)^{(2n)^2}}\right)^{\frac{3}{4}} \times \left(\frac{\prod_{n=1}^{2N} n}{\prod_{n=1}^{N}(2n)}\right)^{\frac{1}{4}}
\end{aligned}$$

and the (generalized) Stirling formulas

$$\prod_{n=1}^{2N} n^{n^2} \sim e^{-\zeta'(-2)}(2N)^{\frac{8N^3}{3}+2N^2+\frac{N}{3}} e^{-\frac{8N^3}{9}+\frac{N}{6}},$$

$$\begin{aligned}
\prod_{n=1}^{N}(2n)^{(2n)^2} &= 2^{4(1^2+\cdots+N^2)}\left(\prod_{n=1}^{N} n^{n^2}\right)^4 \\
&\sim 2^{\frac{2}{3}N(N+1)(2N+1)}\left(e^{-\zeta'(-2)}N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}}e^{-\frac{N^3}{9}+\frac{N}{12}}\right)^4,
\end{aligned}$$

$$\prod_{n=1}^{2N} n \sim \sqrt{2\pi}(2N)^{2N+\frac{1}{2}}e^{-2N},$$

$$\prod_{n=1}^{N}(2n) = 2^N \prod_{n=1}^{N} n \sim \sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}2^N,$$

we have

$$\begin{aligned}
A_N &\sim \exp\left(\frac{1}{24}+\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)+\frac{N}{12}\right) \\
&\quad \times e^{\frac{9}{4}\zeta'(-2)}N^{N^3-\frac{N}{4}}2^{N^3-\frac{N}{4}}e^{-\frac{N^3}{3}-\frac{N}{8}} \\
&\quad \times N^{\frac{N}{4}}2^{\frac{N}{4}+\frac{1}{8}}e^{-\frac{N}{4}} \times (2N+1)^{-N^3}.
\end{aligned}$$

Hence, combining with

$$\begin{aligned}
(2N+1)^{-N^3} &= (2N)^{-N^3}\left(1+\frac{1}{2N}\right)^{-N^3} \\
&= (2N)^{-N^3}\exp\left(-N^3\log\left(1+\frac{1}{2N}\right)\right) \\
&\sim (2N)^{-N^3}\exp\left(-N^3\left(\frac{1}{2N}-\frac{1}{2}\left(\frac{1}{2N}\right)^2+\frac{1}{3}\left(\frac{1}{2N}\right)^3\right)\right) \\
&= (2N)^{-N^3}\exp\left(-\frac{N^2}{2}+\frac{N}{8}-\frac{1}{24}\right),
\end{aligned}$$

we obtain the desired result

$$\lim_{N\to\infty} A_N = 2^{\frac{1}{8}}\exp\left(\frac{9}{4}\zeta'(-2)\right). \blacksquare$$

Remarks

1. The fact $\mathscr{S}_2(\frac{1}{2}) = \sqrt{2}$ is also proved as Theorem 3 and we get (1.1) again from (1.4). In fact:

$$\begin{aligned}
\mathscr{S}_2\left(\frac{1}{2}\right) &= e^{\frac{1}{2}}\prod_{n=1}^{\infty}\left(\left(\frac{1-\frac{1}{2n}}{1+\frac{1}{2n}}\right)^n e\right) \\
&= \lim_{N\to\infty} e^{\frac{1}{2}}\prod_{n=1}^{N}\left(\left(\frac{2n-1}{2n+1}\right)^n e\right) \\
&= \lim_{N\to\infty}\left(e^{\frac{1}{2}}\left(\frac{1}{3}\right)^1\left(\frac{3}{5}\right)^2\left(\frac{5}{7}\right)^3\cdots\left(\frac{2N-1}{2N+1}\right)^N e^N\right) \\
&= \lim_{N\to\infty}\left(e^{\frac{1}{2}}\frac{3\cdot 5\cdots(2N-1)}{(2N+1)^N}e^N\right) \\
&= \lim_{N\to\infty}\left(e^{\frac{1}{2}}\frac{(2N)!}{N!2^{2N}N^N(1+\frac{1}{2N})^N}e^N\right) \\
&= \sqrt{2}
\end{aligned}$$

by the (usual) Stirling formula.

2. The case of $\mathscr{S}_3(1/2)$ is similar by using the generalized Stirling's formula:

$$\begin{aligned}
\mathscr{S}_3\left(\frac{1}{2}\right) &= e^{\frac{1}{8}}\prod_{n=1}^{\infty}\left(\left(1-\frac{1}{4n^2}\right)^{n^2}e^{\frac{1}{4}}\right) \\
&= \lim_{N\to\infty} e^{\frac{1}{8}}\prod_{n=1}^{N}\left(\left(1-\frac{1}{4n^2}\right)^{n^2}e^{\frac{1}{4}}\right) \\
&= \lim_{N\to\infty} e^{\frac{N}{4}+\frac{1}{8}}\prod_{n=1}^{N}\left(\frac{(2n-1)(2n+1)}{(2n)^2}\right)^{n^2} \\
&= \lim_{N\to\infty} e^{\frac{N}{4}+\frac{1}{8}}\frac{\left(\prod_{n=1}^{2N} n^{n^2}\right)^{1/2}}{\left(\prod_{n=1}^{N} n^{n^2}\right)^{4}} \times \left(\frac{(2N)!}{2^N N!}\right)^{\frac{1}{2}} \times \frac{(2N+1)^{N^2}}{2^{4(1^2+\cdots+N^2)}} \\
&= 2^{\frac{1}{4}}\exp\left(\frac{7}{2}\zeta'(-2)\right).
\end{aligned}$$

Thus we obtain Euler's formula (1.2) via (1.5).

3. Except for $\mathscr{S}_2(1/2)=\sqrt{2}$ we do not know the algebraicity of $\mathscr{S}_r(1/2)$ for $r\geq 2$. In fact we cannot deny even the optimistic expectation $\mathscr{S}_r(1/2)\in 2^{\mathbf{Q}}$.

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