Division Values of Multiple Sine Functions

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Abstract. We refine a formula on values of multiple sine functions at division points. As applications we prove a formula on a sum of reciprocal trigonometric values, and obtain multiple modularity of a three variable modular function, which concerns a generalization of the Dedekind η function.

Key words: Multiple sine function; division values; rationality.

1 Introduction

Throughout this paper we put $r \geq 1$ to be an integer and let

$$\mathcal{D}_r = \left\{ \boldsymbol{\omega} = (\omega_1, \cdots, \omega_r) \in \mathbf{C}^r \,\middle|\, \begin{array}{c} \omega_1, \cdots, \omega_r \text{ and } 1 \text{ belong to one side} \\ \text{with respect to a line crossing } 0 \end{array} \right\}.$$

For $\boldsymbol{\omega} \in \mathcal{D}_r$ and $x \in \mathbf{C} \setminus \{-m_1\omega_1 - \cdots - m_r\omega_r \,|\, m_j \in \mathbf{Z}_{\geq 0}\}$, the multiple gamma function is defined by

$$\Gamma_r(x; \boldsymbol{\omega}) = \exp\left(\zeta'(0,\, x;\, \boldsymbol{\omega})\right),$$

where $\zeta_r(s,\, x;\, \boldsymbol{\omega})$ is the multiple Hurwitz zeta function defined by

$$\zeta_r(s,\ x;\ \boldsymbol{\omega}) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (m_1\omega_1 + \cdots + m_r\omega_r + x)^{-s}$$

which is absolutely convergent in $\mathrm{Re}(s) > r$ and has a meromorphic continuation to the entire plane. The multiple sine function

$$S_r(x; \boldsymbol{\omega}) = \Gamma_r(x; \boldsymbol{\omega})^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x; \boldsymbol{\omega})^{(-1)^r}$$

was first introduced by Kurokawa [K1], which was a generalization of Shintani's function $F(x;\, (\omega_1, \omega_2)) := S_2(x;\, (\omega_1, \omega_2))$. In pursueing Kronecker's Jugendtraum for a real quadratic field, Shintani [S] introduced $F(x;\,(\omega_1,\omega_2))$ as a generalization of the sine function $S_1(x) := S_1(x;\,1) = 2\sin(\pi x)$. More precisely, for an L-function $L_K(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})N(\mathfrak{a})^{-s}$ for a real quadratic field K, he expressed the value

$$\exp(-L_K'(0,\chi))$$

in terms of a product of division values of $S_2(x;\,(1,\varepsilon))$ with ε the fundamental unit of K. This is regarded as a generalization of a well-known formula over $\mathbf{Q}$:

$$\exp(-L'(0,\chi)) \;=\; \prod_{k=1}^{N-1} S_1\left(\frac{k}{N}\right)^{\chi(k)/2}, \tag{1.1}$$

where $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ is the Dirichle L-function for a non-trivial primitive even character χ modulo N. The formula (1.1) gives a reason for the appearance of the sine function in Kronecker's Jugendtraum over $\mathbf{Q}$ in the sense that the division values of $S_1(x)$ generate the maximal abelian extension of $\mathbf{Q}$. Discovering a function with an analogous property for totally real number fields is an open unsolved problem. Unfortunately very few sequent works are seen after Shintani's pioneering work [S] in 1977.

The fundamental importance of division values of multiple sine functions is also seen from another aspect concerning special values of zeta functions. Indeed there is another direction for the extension of the formula (1.1). In a previous paper [KK2] we gave a generalization of (1.1) to $s \neq 0$. In this generalization, we fix the period to be trivial as we consider over $\mathbf{Q}$, and simply denote $S_r(x) := S_r(x,\,(1,\cdots,1))$. In the case of $N=1$, we proved for $n = 1,2,3,\cdots$ that

$$\exp(-\zeta'(-2n)) = \prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}$$

with explicitly determined positive integers $a(2n+1,k)$, and we also proved for χ being primitive odd modulo N that

$$\exp(-L'(-1,\chi)) = \prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^{N} S_1\left(\frac{k}{N}\right)^{k}\right)^{\chi(k)/2},$$

and for χ non-trivial primitive even modulo N that

$$\exp(-L'(-2,\chi)) = \prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)/2}.$$

These enable to express unknown special values such as

$$\begin{aligned}
\zeta(3) &= 4\pi^2 \log S_3(1), \\
\zeta(5) &= -\frac{4\pi^4}{3} \log(S_5(1)S_5(2)^{11}), \\
\zeta(7) &= \frac{8\pi^6}{45} \log(S_7(1)S_7(2)^{57}S_7(3)^{302}), \\
L(2, \left(\frac{-4}{*}\right)) &= -\frac{\pi}{4} \log \left(S_2\left(\frac{1}{4}\right)^4 S_1\left(\frac{1}{4}\right) S_2\left(\frac{3}{4}\right)^{-4} S_1\left(\frac{3}{4}\right)^{-3} \right) \\
&= \frac{\pi}{4} \log \left(2^3 S_2\left(\frac{1}{4}\right)^{-8} \right), \\
L(3, \left(\frac{12}{*}\right)) &= \frac{\sqrt{3}\pi^2}{432} \log \left(S_3\left(\frac{1}{12}\right)^{288} S_2\left(\frac{1}{12}\right)^{-408} S_1\left(\frac{1}{12}\right) S_3\left(\frac{5}{12}\right)^{-288} S_2\left(\frac{5}{12}\right)^{312} S_1\left(\frac{5}{12}\right)^{-25} \right. \\
& \left. S_3\left(\frac{7}{12}\right)^{-288} S_2\left(\frac{7}{12}\right)^{264} S_1\left(\frac{7}{12}\right)^{-49} S_3\left(\frac{11}{12}\right)^{288} S_2\left(\frac{11}{12}\right)^{-164} S_1\left(\frac{11}{12}\right)^{121} \right).
\end{aligned}$$

As shown in these two aspects concerning generalizations of the Dirichlet class number formula (1.1), the study of division values of multiple sine functions seems to be of fundamental importance. Above all, rationality at division points is of central interest.

Fix once and for all an integer $N \geq 2$. A basic relevant fact concerning the rationality of division values is the following formula proved in [KK1]:

$$\prod_{0 \leq k_1, \ldots, k_r \leq N-1}{}' S_r\left(\frac{k_1\omega_1 + \cdots + k_r\omega_r}{N};\ \boldsymbol{\omega}\right) = N. \tag{1.2}$$

(We remark that in [KK1] everything is done for $\omega_j > 0$, but the formula (1.2) holds for $\boldsymbol{\omega} \in \mathcal{D}_r$ by the same proof.) It would be significant to make (1.2) more precise to get more detailed information by determining values of various partial products of the left hand side of (1.2). Kurokawa [K2] recently discovered a beautiful refinement of it for $r = 2$. He proved that the contributions from terms with $k_j = 0$ $(j = 1, 2)$ are equal. Namely, he showed that

$$\prod_{k_1=1}^{N-1} S_2\left(\frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right) = \prod_{k_2=1}^{N-1} S_2\left(\frac{k_2\omega_2}{N};\ \boldsymbol{\omega}\right) = \sqrt{N} \tag{1.3}$$

and that

$$\prod_{1 \leq k_1, k_2 \leq N-1} S_2\left(\frac{k_1\omega_1 + k_2\omega_2}{N};\ \boldsymbol{\omega}\right) = 1. \tag{1.4}$$

He also obtained some applications which will be explained later.

In this paper we first generalize (1.3) and (1.4) to the case of $r = 4$. A generalization of (1.4) is easily described. We show the following theorem (Theorem 4.4 below):

Theorem 1.1

$$\prod_{1\le k_1,k_2,k_3,k_4\le N-1} S_4\left(\frac{k_1\omega_1+\cdots+k_4\omega_4}{N};\ \boldsymbol{\omega}\right)=1. \tag{1.5}$$

One may wonder what happens when $r=3$. Since the situation essentially depends on the parity of r, the formula like (1.4) and (1.5) is not true for r being odd. Actually in Theorem 3.3 below, we obtain an alternative expression for $r=3$ in terms of the division values of the triple gamma function.

For describing a generalization of (1.3) to $r=4$, we start by defining the contribution of $k_j=0$ to the product (1.2). Let A_j be the partial product of (1.2) over terms with $k_j=0$ and $k_l\neq 0$ for $l\neq j$. We similarly denote by $A_{j,l}$ and $A_{j,l,m}$ the partial products of (1.2) over terms with $k_j=k_l=0$ and $k_j=k_l=k_m=0$, respectively, where all other coefficients are nonzero. Then by (1.2) and (1.5) we see for $r=4$ that

$$\left(\prod_{j=1}^{4} A_j\right)\left(\prod_{1\le j<l\le 3} A_{j,l}\right)\left(\prod_{1\le j<l<m\le 3} A_{j,l,m}\right)=N.$$

One of our goals is to determine the contribution from $k_j=0$ for each $j=1,2,3,4$ to (1.2). The key idea is that the contribution of $k_j=0$ from $A_{j,l}$ to (1.2) should be regarded as $(A_{j,l})^{\frac{1}{2}}$, since the product $A_{j,l}$ is shared by $k_j=0$ and $k_l=0$. This idea is also applicable for $A_{j,l,m}$, which is identified as the product of the three contributions from $k_j=0$, $k_l=0$ and $k_m=0$. In this way we reach the following definition:

$$\widetilde{A}_4 := A_4\left(\prod_{j=1}^{3} A_{j,4}\right)^{\frac{1}{2}}\left(\prod_{1\le i<j\le 3} A_{i,j,4}\right)^{\frac{1}{3}}, \tag{1.6}$$

which we call the contribution of terms with $k_4=0$ to the product (1.2). We similarly define $\widetilde{A}_j$ for $j=1,2,3,4$. Taking (1.5) into account, the formula (1.2) is equivalently written as

$$\prod_{j=1}^{4} \widetilde{A}_j = N. \tag{1.7}$$

One of the purposes of this paper is to determine each contribution $\widetilde{A}_j$ in (1.7). Our main theorem is as follows. Without loss of generality we only consider the case $j=4$.

Theorem 1.2 *The contribution from terms with $k_4 = 0$ to the product* (1.2) *is given by*

$$\widetilde{A}_4 = N^{\frac{1}{4}+\frac{1}{72}\left(N\sum_{i\neq j}\left(\frac{\omega_4}{\omega_j}-\frac{\omega_i}{\omega_j}\right)+\frac{1}{N}\sum_{i\neq j}\left(\frac{\omega_j}{\omega_4}-\frac{\omega_i}{\omega_j}\right)\right)}\prod_{i\neq j}\prod_{0\leq k_1,k_2\leq N-1}{}' \left(\frac{\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)}{\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)}\right)^{\frac{1}{2}}$$
$$\times\prod_{k=1}^{N-1}\left(\frac{\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)}{\Gamma_3\left(\frac{k\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)}\right)^{\frac{2}{3}}\left(\frac{\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)}{\Gamma_2\left(\frac{k\omega_j}{N};(\omega_j,\omega_4)\right)}\right)^{\frac{1}{6}},$$

where the sums and the product over $i \neq j$ are taken over all ordered pairs $(i,j) \in \{1,2,3\}^2$ with $i \neq j$.

Corollary 1 *If $\omega_1 = \omega_2 = \omega_3 = \omega_4$, then it holds that*

$$\widetilde{A}_j = N^{\frac{1}{4}} \qquad (j = 1,2,3,4).$$

This shows that Theorem 1.1 is a generalization of (1.3).

Example. When $N = 2$ and $\boldsymbol{\omega} = (1,1,1,1)$, we can compute each factor of $\widetilde{A}_4$ by using the results proved in later sections as follows:

$$\begin{aligned}
A_4 &= S_3\left(\frac{3}{2}\right)^{\frac{1}{2}} = 2^{-\frac{1}{16}}e^{\frac{3}{8}\zeta'(-2)},\\
\left(\prod_{j=1}^{3}A_{j,4}\right)^{\frac{1}{2}} &= S_3(1)^{\frac{3}{2}} = e^{-\frac{3}{2}\zeta'(-2)},\\
\left(\prod_{1\leq i<j\leq 3}A_{i,j,4}\right)^{\frac{1}{3}} &= 2^{-\frac{1}{4}}S_3\left(\frac{1}{2}\right)^{\frac{3}{2}} = 2^{\frac{5}{16}}e^{\frac{9}{8}\zeta'(-2)}.
\end{aligned}$$

By multiplying the both sides we have

$$\widetilde{A}_4 = 2^{\frac{1}{4}}.$$

Division values of multiple sine functions have modular interpretations as indicated in [K2]. Let $\mathrm{Im}(\tau) > 0$ and $q = e^{2\pi i\tau}$. Kurokawa showed that by putting

$$\begin{aligned}
F(\tau) &= q^{-\frac{1}{48}}\prod_{n=0}^{\infty}\left(1+q^{n+\frac{1}{2}}\right)\\
&= \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)}
\end{aligned}$$

with the Dedekind η function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

it holds that

$$F\left(-\frac{1}{\tau}\right) = F(\tau) \tag{1.8}$$

and that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi n \tau)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi n \tau^{-1})} = -\frac{\pi}{12} \left(\tau + \frac{1}{\tau}\right). \tag{1.9}$$

Indeed he showed that both (1.8) and (1.9) are equivalent to (1.4) for $N = 2$ and $(\omega_1, \omega_2) = (1, \tau)$. In Section 7 we generalize such modular interpretations to $r = 4$.

The first result is a generalization of (1.9).

Theorem 1.3 *Let* $0 < \arg(\tau_3) < \arg(\tau_2) < \arg(\tau_1) < \pi$ *and* $0 < \mathrm{Re}(\tau_1 + \tau_2 + \tau_3 + 1)$. *Then it holds that*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi n \tau_1) \sin(\pi n \tau_2) \sin(\pi n \tau_3)} \\
&= 8\pi\zeta_4\left(0, \frac{\tau_1 + \tau_2 + \tau_3 + 1}{2}, (\tau_1, \tau_2, \tau_3, 1)\right) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi \frac{n}{\tau_1}) \sin(\pi \frac{n\tau_2}{\tau_1}) \sin(\pi \frac{n\tau_3}{\tau_1})} \\
&\quad - \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi \frac{n\tau_1}{\tau_2}) \sin(\pi \frac{n}{\tau_2}) \sin(\pi \frac{n\tau_3}{\tau_2})} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(\pi \frac{n\tau_1}{\tau_3}) \sin(\pi \frac{n\tau_2}{\tau_3}) \sin(\pi \frac{n}{\tau_3})},
\end{aligned}$$

where $\zeta_4\left(0, \frac{\tau_1+\tau_2+\tau_3+1}{2}, (\tau_1, \tau_2, \tau_3, 1)\right)$*, the value of the multiple Hurwitz zeta function, is explicitly given in terms of* τ_j $(j = 1, 2, 3)$ *in Lemma 7.1.*

In the following theorem we obtain multiple modularity of the three variable function

$$F(\tau_1, \tau_2, \tau_3) = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \prod_{n_3=0}^{\infty} \left(1 + q_1^{n_1+\frac{1}{2}} q_2^{n_2+\frac{1}{2}} q_3^{n_3+\frac{1}{2}}\right)$$

for $0 < \arg(\tau_3) < \arg(\tau_2) < \arg(\tau_1) < \pi$, $0 < \mathrm{Re}(\tau_1 + \tau_2 + \tau_3 + 1)$ and $q_j = e^{2\pi i \tau_j}$ with $j = 1, 2, 3$.

Theorem 1.4 *Put* $x = (\tau_1 + \tau_2 + \tau_3 + 1)/2$ *and* $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, 1)$. *Then we have the following multiple modularity:*

$$F(\tau_1, \tau_2, \tau_3) = e^{\pi i \zeta_4(0, x; \boldsymbol{\tau})} \frac{F\left(-\frac{1}{\tau_1}, -\frac{\tau_2}{\tau_1}, -\frac{\tau_3}{\tau_1}\right) F\left(\frac{\tau_1}{\tau_3}, \frac{\tau_2}{\tau_3}, -\frac{1}{\tau_3}\right)}{F\left(\frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2}, -\frac{\tau_3}{\tau_2}\right)},$$

where $\zeta_4(0, x; \boldsymbol{\tau})$, the value of the multiple Hurwitz zeta function, is explicitly given in terms of τ_j $(j = 1, 2, 3)$ in Lemma 7.1.

Finally in Section 8 we investigate the behavior of the multiple sine function for real positive periods $\boldsymbol{\omega} \in \mathbf{R}^r_{>0}$, and locate some two-division points. In the previous paper [KK3] we draw a graph of $S_2(x; (\omega_1, \omega_2))$, and more recently Kurokawa [K3] did it for $S_3(x; (\omega_1, \omega_2, \omega_3))$. We present a generalization of these works to $r = 4$. We will show in Theorem 8.1 that $S_4(x, \boldsymbol{\omega})$ has four extremal values in the fundamental period $(0, |\boldsymbol{\omega}|)$ with $|\boldsymbol{\omega}| = \omega_1 + \omega_2 + \omega_3 + \omega_4$, and that each interval of $(0, |\boldsymbol{\omega}|/2)$ and $(|\boldsymbol{\omega}|/2, |\boldsymbol{\omega}|)$ has both a maximal and a minimal values. The behavior of $S_4(x, \boldsymbol{\omega})$ is roughly shown in Figure 1.

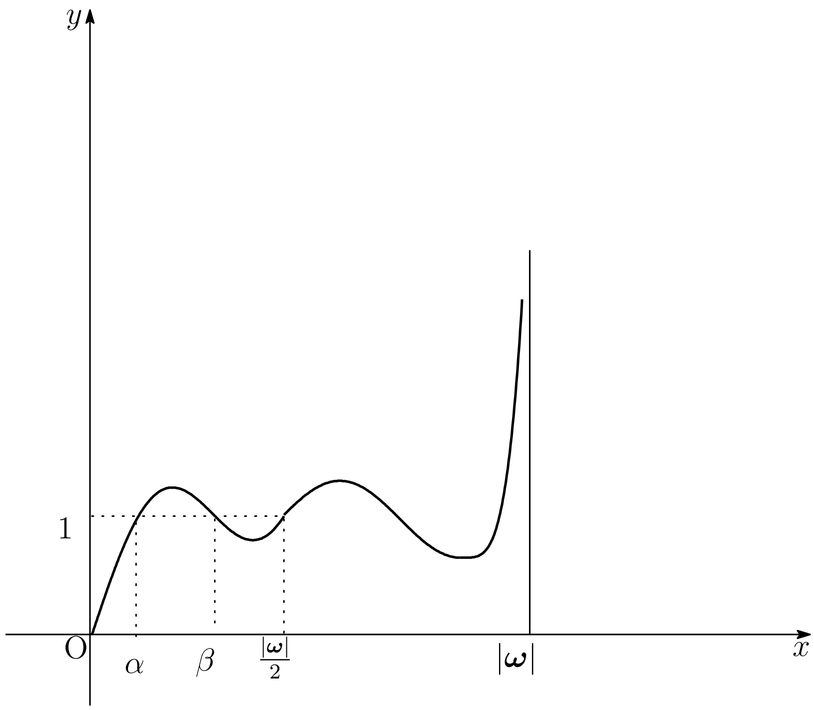

Figure 1. The graph of $S(x) = S_4(x; \boldsymbol{\omega})$.

We will see in Theorem 8.2 that there exist 2-division points both in (α, β) and $(\beta, |\boldsymbol{\omega}|/2)$.

Thoughout the proof we use the following notation.

Notation. For $r = 3$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, we put for any integer N

$$\boldsymbol{\omega}_1^N = (\omega_1, N\omega_2, N\omega_3), \qquad \boldsymbol{\omega}_2^N = (N\omega_1, \omega_2, N\omega_3), \qquad \boldsymbol{\omega}_3^N = (N\omega_1, N\omega_2, \omega_3),$$
$$\boldsymbol{\omega}_1^{(N)} = (N\omega_1, \omega_2, \omega_3), \qquad \boldsymbol{\omega}_2^{(N)} = (\omega_1, N\omega_2, \omega_3), \qquad \boldsymbol{\omega}_3^{(N)} = (\omega_1, \omega_2, N\omega_3),$$

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2 Stirling Modular Forms

The Stirling modular form is defined as

$$\rho_r(\boldsymbol{\omega})^{-1} = \lim_{x\to 0} x\Gamma(x;\ \boldsymbol{\omega}).$$

We introduce an auxiliary zeta function as

$$\zeta_r(s;\ \boldsymbol{\omega}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} (m_1\omega_1 + \cdots m_r\omega_r)^{-s}.$$

It is related to $\zeta_r(s,\ x;\ \boldsymbol{\omega})$ as

$$\begin{aligned}&\lim_{x\to 0}\left(\zeta_r(s,\ x;\ \boldsymbol{\omega}) - x^{-s}\right)\\ &= \zeta_r(s;\ \boldsymbol{\omega}) + \sum_{j=1}^{r} \zeta_{r-1}(s;\ \boldsymbol{\omega}(j)) + \sum_{\substack{1\le j,k\le r\\ j<k}} \zeta_{r-2}(s;\ \boldsymbol{\omega}(j,k)) + \cdots + \left(\omega_1^{-s} + \cdots + \omega_r^{-s}\right)\zeta(s).\end{aligned} \tag{2.1}$$

Proposition 2.1

$$\zeta_2(0;\ (\omega_1,\omega_2)) = \frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2} = \frac{1}{12}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+3\right),$$

$$\zeta_3(0;\ \boldsymbol{\omega}) = -\frac{1}{24}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_3}+\frac{\omega_3}{\omega_1}+\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_3}+\frac{\omega_3}{\omega_2}+3\right) = -\frac{1}{24}\sum_{i,\ j}\frac{\omega_i}{\omega_j},$$

where the sum is taken over all ordered pairs (i,j) *with* $i,j\in\{1,2,3\}$.

Proof. We first show the following two identities in turn.

$$\lim_{x\to 0}\zeta_2(0,\ x;\ (\omega_1,\omega_2)) = \frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2} = \frac{1}{12}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+3\right),$$

$$\lim_{x\to 0}\zeta_3(0,\ x;\ \boldsymbol{\omega}) = \frac{1}{24}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_3}+\frac{\omega_3}{\omega_1}+\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_3}+\frac{\omega_3}{\omega_2}+3\right).$$

We appeal to an integral representation

$$\lim_{x\to 0}\zeta_2(s,\ x;\ \boldsymbol{\omega}) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-t)^{s-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}dt,$$

where C is the standard coutour consisting of $+\infty \to \varepsilon > 0, \quad \varepsilon e^{i\theta}\ (0 \le \theta \le 2\pi), \quad \varepsilon \to +\infty$. By the expansion given by [B], we have

$$\begin{aligned}
\lim_{x\to 0} \zeta_2(0,\ x;\ \boldsymbol{\omega}) &= \frac{1}{2\pi i}\int_C \frac{t^{-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}dt \\
&= \frac{1}{2\pi i}\int_C \left(\frac{{}_2S_1^{(3)}(0)}{t} - {}_2S_1^{(2)}(0) + {}_2S_1'(0)t + \cdots \right)\frac{dt}{t^2} \\
&= {}_2S_1'(0) \\
&= \frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2}.
\end{aligned}$$

For the second identity by putting

$$\begin{aligned}
f(t) &= \frac{t}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})} \\
&= \frac{a}{t^2} + \frac{b}{t} + c + dt + et^2 + \cdots,
\end{aligned}$$

we compute

$$\begin{aligned}
\lim_{x\to 0} \zeta_3(0,\ x;\ \boldsymbol{\omega}) &= \frac{1}{2\pi i}\int_C \frac{t^{-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})}dt \\
&= \frac{1}{2\pi i}\int_C \frac{f(t)}{t^2}dt = d.
\end{aligned}$$

We calculate each coefficient in order as follows:

$$\begin{aligned}
a &= \frac{1}{\omega_1\omega_2\omega_3}, \\
b &= \frac{\omega_1+\omega_2+\omega_3}{2\omega_1\omega_2\omega_3}, \\
c &= \frac{\omega_1^2+\omega_2^2+\omega_3^2+3(\omega_1\omega_2+\omega_2\omega_3+\omega_3\omega_1)}{12\omega_1\omega_2\omega_3}, \\
d &= \frac{1}{24}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_1} + \frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_3} + \frac{\omega_3}{\omega_2} + 3 \right).
\end{aligned}$$

(Under the notation in [B], we write

$$a = {}_3S_1^{(4)}(0), \qquad b = -\,{}_3S_1^{(3)}(0), \qquad c = {}_3S_1^{(2)}(0), \qquad d = -\,{}_3S_1'(0), \qquad e = \frac{{}_3S_2'(0)}{2}.)$$

Hence we obtain the two values $\lim_{x\to 0}\zeta_2(0,\ x;\ \boldsymbol{\omega})$ and $\lim_{x\to 0}\zeta_3(0,\ x;\ \boldsymbol{\omega})$.

From (2.1) it follows that

$$
\begin{aligned}
\lim_{x\to 0}\zeta_2(0,x;\boldsymbol{\omega}) &= 1+\zeta_2(0;\boldsymbol{\omega})+\zeta_1(0,\omega_1)+\zeta_1(0,\omega_2)\\
&= 1+\zeta_2(0;\boldsymbol{\omega})+\left(\omega_1^{-s}+\omega_2^{-s}\right)_{s=0}\zeta(0)\\
&= \zeta_2(0;\boldsymbol{\omega}),
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{x\to 0}\zeta_3(0,x;\boldsymbol{\omega}) &= 1+\zeta_3(0;\boldsymbol{\omega})+\sum_{\substack{1\le i,j\le 3\\ i\ne j}}\zeta_2(0;(\omega_i,\omega_j))+\sum_{j=1}^{3}\zeta_1(0,\omega_j)\\
&= 1+\zeta_3(0;\boldsymbol{\omega})+\frac{1}{12}\left(\sum_{\substack{1\le i,j\le 3\\ i\ne j}}\frac{\omega_i}{\omega_j}+9\right)-\frac{3}{2}\\
&= 1+\zeta_3(0;\boldsymbol{\omega})+\frac{1}{12}\left(\sum_{1\le i,j\le 3}\frac{\omega_i}{\omega_j}+6\right)-\frac{3}{2}\\
&= \zeta_3(0;\boldsymbol{\omega})+\frac{1}{12}\sum_{1\le i,j\le 3}\frac{\omega_i}{\omega_j}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\zeta_3(0;\boldsymbol{\omega}) &= \frac{1}{12}\sum_{1\le i,j\le 3}\frac{\omega_i}{\omega_j}-\lim_{x\to 0}\zeta_3(0,x;\boldsymbol{\omega})\\
&= -\frac{1}{24}\sum_{1\le i,j\le 3}\frac{\omega_i}{\omega_j}
\end{aligned}
$$

as desired. ∎

Proposition 2.2

$$
\begin{aligned}
\zeta_2'(0;\ (\omega_1,\omega_2)) &= \log\frac{2\pi}{\rho_2(\omega_1,\omega_2)\sqrt{\omega_1\omega_2}},\\
\zeta_3'(0;\ \boldsymbol{\omega}) &= \log\frac{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_3,\omega_1)\sqrt{\omega_1\omega_2\omega_3}}{(2\pi)^{\frac{3}{2}}\rho_3(\boldsymbol{\omega})}.
\end{aligned}
$$

Proof. By definition we have

$$
\begin{aligned}
&\zeta_2(s,\ x;\ (\omega_1,\omega_2))\\
&= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}(m_1\omega_1+m_2\omega_2+x)^{-s}\\
&= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}(m_1\omega_1+m_2\omega_2+x)^{-s}+\sum_{m_1=1}^{\infty}(m_1\omega_1+x)^{-s}+\sum_{m_2=1}^{\infty}(m_2\omega_2+x)^{-s}+x^{-s}
\end{aligned}
$$

and thus

$$\lim_{x\to 0}\frac{d}{ds}\left(\zeta_2(s,\ x;\ (\omega_1,\omega_2))-x^{-s}\right)\bigg|_{s=0}=\zeta_2'(0;\ (\omega_1,\omega_2))+2\zeta'(0)-(\log\omega_1\omega_2)\zeta(0).$$

As $\zeta'(0)=-\frac{\log 2\pi}{2}$ and $\zeta(0)=-\frac{1}{2}$, we have

$$\begin{aligned}
\zeta_2'(0;\ (\omega_1,\omega_2)) &= \log 2\pi-\frac{\log\omega_1\omega_2}{2}+\lim_{x\to 0}\left(\zeta_2'(0,\ x;\ (\omega_1,\omega_2))+\log x\right)\\
&= \log 2\pi-\frac{\log\omega_1\omega_2}{2}+\lim_{x\to 0}\left(\log\Gamma_2(x;\ (\omega_1,\omega_2))+\log x\right)\\
&= \log 2\pi-\frac{\log\omega_1\omega_2}{2}+\lim_{x\to 0}\log(x\Gamma_2(x;\ (\omega_1,\omega_2)))\\
&= \log 2\pi-\frac{\log\omega_1\omega_2}{2}-\log\rho_2(\omega_1,\omega_2)\\
&= \log\frac{2\pi}{\rho_2(\omega_1,\omega_2)\sqrt{\omega_1\omega_2}}.
\end{aligned}$$

For the second identity we have

$$\begin{aligned}
&\zeta_3(s,\ x;\ \boldsymbol{\omega})\\
&\quad=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}(m_1\omega_1+m_2\omega_2+m_3\omega_3+x)^{-s}\\
&\quad=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}(m_1\omega_1+m_2\omega_2+m_3\omega_3+x)^{-s}+\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}(m_1\omega_1+m_2\omega_2+x)^{-s}\\
&\qquad+\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}(m_2\omega_2+m_3\omega_3+x)^{-s}+\sum_{m_1=1}^{\infty}\sum_{m_3=1}^{\infty}(m_1\omega_1+m_3\omega_3+x)^{-s}\\
&\qquad+\sum_{m_1=1}^{\infty}(m_1\omega_1+x)^{-s}+\sum_{m_2=1}^{\infty}(m_2\omega_2+x)^{-s}+\sum_{m_3=1}^{\infty}(m_3\omega_3+x)^{-s}+x^{-s},
\end{aligned}$$

and thus

$$\begin{aligned}
&\lim_{x\to 0}\frac{d}{ds}\left(\zeta_3(s,\ x;\ \boldsymbol{\omega})-x^{-s}\right)\bigg|_{s=0}\\
&\quad=\zeta_3'(0;\ \boldsymbol{\omega})+\zeta_2'(0;\ (\omega_1,\omega_2))+\zeta_2'(0;\ (\omega_2,\omega_3))+\zeta_2'(0;\ (\omega_1,\omega_3))+3\zeta'(0)-(\log\omega_1\omega_2\omega_3)\zeta(0)\\
&\quad=\zeta_3'(0;\ \boldsymbol{\omega})+\log\frac{(2\pi)^3}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)\omega_1\omega_2\omega_3}-\frac{3}{2}\log 2\pi+\frac{\log\omega_1\omega_2\omega_3}{2}\\
&\quad=\zeta_3'(0;\ \boldsymbol{\omega})+\log\frac{(2\pi)^{\frac{3}{2}}}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)\sqrt{\omega_1\omega_2\omega_3}}.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
\zeta_3'(0;\ \boldsymbol{\omega}) &= \log \frac{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)\sqrt{\omega_1\omega_2\omega_3}}{(2\pi)^{\frac{3}{2}}} + \lim_{x\to 0}\left(\zeta_3'(0,\ x;\ \boldsymbol{\omega}) + \log x\right) \\
&= \log \frac{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)\sqrt{\omega_1\omega_2\omega_3}}{(2\pi)^{\frac{3}{2}}} + \lim_{x\to 0}\log(x\Gamma_3(x;\ \boldsymbol{\omega})) \\
&= \log \frac{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)\sqrt{\omega_1\omega_2\omega_3}}{(2\pi)^{\frac{3}{2}}\rho_3(\boldsymbol{\omega})}.
\end{aligned}
$$

∎

Lemma 2.1

$$
\begin{aligned}
\rho_r(N\boldsymbol{\omega})^{-1} &= \rho_r(\boldsymbol{\omega})^{-1}N^{1-\lim_{x\to 0}\zeta_r(0,x;\boldsymbol{\omega})} \\
&= \rho_r(\boldsymbol{\omega})^{-1} \times \begin{cases} N^{1-\frac{1}{12}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+3\right)} & (r=2) \\ N^{1-\frac{1}{24}\sum\limits_{1\le i,j\le 3}\frac{\omega_i}{\omega_j}}. & (r=3) \end{cases}
\end{aligned}
$$

Proof. We compute that

$$
\begin{aligned}
\rho_r(N\boldsymbol{\omega})^{-1} &= \lim_{x\to 0} x\Gamma_r(x, N\boldsymbol{\omega}) \\
&= \lim_{x\to 0} x\exp\left(\zeta_r'(0, x, N\boldsymbol{\omega})\right) \\
&= \lim_{x\to 0} x\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{\mathbf{m}\ge 0}(\mathbf{m}\cdot(N\boldsymbol{\omega})+x)^{-s}\right) \\
&= \lim_{x\to 0} x\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0} N^{-s}\sum_{\mathbf{m}\ge 0}(\mathbf{m}\cdot\boldsymbol{\omega}+\frac{x}{N})^{-s}\right) \\
&= \lim_{x\to 0} x\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0} N^{-s}\zeta_r(s,\frac{x}{N};\boldsymbol{\omega})\right) \\
&= \lim_{x\to 0} x\exp\left(N^{-s}(-\log N)\zeta_r(s,\frac{x}{N};\boldsymbol{\omega}) + N^{-s}\zeta_r'(s,\frac{x}{N};\boldsymbol{\omega})\right)_{s=0} \\
&= \lim_{x\to 0} x\exp\left((-\log N)\zeta_r(0,\frac{x}{N};\boldsymbol{\omega}) + \zeta_r'(0,\frac{x}{N};\boldsymbol{\omega})\right) \\
&= \lim_{x\to 0} \frac{x}{N}\Gamma_r\left(\frac{x}{N};\boldsymbol{\omega}\right)N^{1-\zeta_r(0,\frac{x}{N};\boldsymbol{\omega})} \\
&= \rho_r(\boldsymbol{\omega})^{-1}N^{1-\lim_{x\to 0}\zeta_r(0,x;\boldsymbol{\omega})}.
\end{aligned}
$$

The values $\lim_{x\to 0}\zeta_r(0,x;\boldsymbol{\omega})$ for $r=2,3$ are calculated in the proof of Proposition 2.1. ∎.

Lemma 2.2

$$
\frac{\rho_r(\boldsymbol{\omega})}{\rho_r\left(\frac{\omega_1}{N_1},\ldots,\frac{\omega_r}{N_r}\right)} = \prod_{k_1=1}^{N_1-1}\cdots\prod_{k_r=1}^{N_r-1}\Gamma_r\left(\frac{k_1\omega_1}{N_1}+\cdots+\frac{k_r\omega_r}{N_r};\boldsymbol{\omega}\right).
$$

Proof.

$$\begin{aligned}
\frac{\rho_r(\boldsymbol{\omega})}{\rho_r(\frac{\omega_1}{N_1},...,\frac{\omega_r}{N_r})} &= \lim_{x\to 0}\frac{\Gamma_2(x;(\frac{\omega_1}{N_1},...,\frac{\omega_r}{N_r}))}{\Gamma_2(x;\boldsymbol{\omega})} \\
&= \lim_{x\to 0}\exp\left(\zeta_r'\left(0,x,\left(\frac{\omega_1}{N_1},...,\frac{\omega_r}{N_r}\right)\right)-\zeta_r'(0,x;\boldsymbol{\omega})\right).
\end{aligned}$$

Here we compute

$$\begin{aligned}
&\zeta_r\left(s,x,\left(\frac{\omega_1}{N_1},...,\frac{\omega_r}{N_r}\right)\right)-\zeta_r(s,x;\boldsymbol{\omega}) \\
= &\sum_{\mathbf{m}\geq 0}\left(\left(\frac{m_1\omega_1}{N_1}+\cdots+\frac{m_r\omega_r}{N_r}+x\right)^{-s}-(\mathbf{m}\cdot\boldsymbol{\omega}+x)^{-s}\right) \\
= &\sum_{\mathbf{m}\geq 0}\left(\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_r=0}^{N_r-1}\left(\frac{(m_1N_1+k_1)\omega_1}{N_1}+\cdots+\frac{(m_rN_r+k_r)\omega_r}{N_r}+x\right)^{-s}\right)-\sum_{\mathbf{m}\geq 0}(\mathbf{m}\cdot\boldsymbol{\omega}+x)^{-s} \\
= &\sum_{\mathbf{m}\geq 0}\sum_{k_1=1}^{N_1-1}\cdots\sum_{k_r=1}^{N_r-1}\left(\frac{(m_1N_1+k_1)\omega_1}{N_1}+\cdots+\frac{(m_rN_r+k_r)\omega_r}{N_r}+x\right)^{-s} \\
= &\sum_{\mathbf{m}\geq 0}\sum_{k_1=1}^{N_1-1}\cdots\sum_{k_r=1}^{N_r-1}\left(\mathbf{m}\cdot\boldsymbol{\omega}+\frac{k_1\omega_1}{N_1}+\cdots+\frac{k_r\omega_r}{N_r}+x\right)^{-s} \\
= &\sum_{k_1=1}^{N_1-1}\cdots\sum_{k_r=1}^{N_r-1}\zeta_r\left(s,\frac{k_1\omega_1}{N_1}+\cdots+\frac{k_r\omega_r}{N_r}+x;\boldsymbol{\omega}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\rho_r(\boldsymbol{\omega})}{\rho_r(\frac{\omega_1}{N_1},...,\frac{\omega_r}{N_r})} &= \lim_{x\to 0}\prod_{k_1=1}^{N_1-1}\cdots\prod_{k_r=1}^{N_r-1}\exp\left(\zeta_r'\left(0,\frac{k_1\omega_1}{N_1}+\cdots+\frac{k_r\omega_r}{N_r}+x;\boldsymbol{\omega}\right)\right) \\
&= \prod_{k_1=1}^{N_1-1}\cdots\prod_{k_r=1}^{N_r-1}\Gamma_r\left(\frac{k_1\omega_1}{N_1}+\cdots+\frac{k_r\omega_r}{N_r};\boldsymbol{\omega}\right).
\end{aligned}$$

∎

Lemma 2.3

$$\frac{\rho_2(\omega_1,\omega_2)}{\rho_2(N\omega_1,\omega_2)}=N^{\frac{1}{12}\left(9-\left(\frac{\omega_1 N}{\omega_2}+\frac{\omega_2}{\omega_1 N}\right)\right)}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_2}{N};(\omega_1,\omega_2)\right).$$

Proof. We apply Lemma 2.2 to the case $r = 2$, $N_1 = 1$, $N_2 = N$. Then

$$\frac{\rho_2(\omega_1, \omega_2)}{\rho_2(\omega_1, \frac{\omega_2}{N})} = \prod_{k=1}^{N-1} \Gamma_2\left(\frac{k\omega_2}{N}; (\omega_1, \omega_2)\right).$$

By Lemma 2.1

$$\begin{aligned}\rho_2(\omega_1, \frac{\omega_2}{N}) &= \rho_2(N\omega_1, \omega_2) N^{1-\frac{1}{12}\left(\frac{\omega_1 N}{\omega_2}+\frac{\omega_2}{\omega_1 N}+3\right)} \\ &= \rho_2(N\omega_1, \omega_2) N^{\frac{1}{12}\left(9-\left(\frac{\omega_1 N}{\omega_2}+\frac{\omega_2}{\omega_1 N}\right)\right)}.\end{aligned}$$

Hence we have the conclusion. ∎

Lemma 2.4 *(a) For a sequence of positive numbers ω_j $(j = 1, 2, 3)$, we have*

$$\prod_{\substack{1\le i,j\le 3\\ i\ne j}} \frac{\rho_2(\omega_i, \omega_j)}{\rho_2(N\omega_i, \omega_j)} = N^{\frac{9}{2}-\frac{1}{12}\left(N+\frac{1}{N}\right)\sum\limits_{i\ne j}\frac{\omega_i}{\omega_j}} \prod_{\substack{1\le i,j\le 3\\ i\ne j}} \prod_{k=1}^{N-1} \Gamma_2\left(\frac{k\omega_j}{N}; (\omega_i, \omega_j)\right).$$

(b) For a sequence of positive numbers ω_j $(j = 1, 2, 3, 4)$, we have

$$\prod_{\substack{1\le i,j\le 4\\ i\ne j}} \frac{\rho_2(\omega_i, \omega_j)}{\rho_2(N\omega_i, \omega_j)} = N^{9-\frac{1}{12}\left(N+\frac{1}{N}\right)\sum\limits_{i\ne j}\frac{\omega_i}{\omega_j}} \prod_{\substack{1\le i,j\le 4\\ i\ne j}} \prod_{k=1}^{N-1} \Gamma_2\left(\frac{k\omega_j}{N}; (\omega_i, \omega_j)\right),$$

where in the right hand side the sum over $i \ne j$ is taken over all ordered pairs $(i, j) \in \{1, 2, 3, 4\}^2$ with $i \ne j$.

Proof. We take a product of the result of the previous lemma over all ordered pairs (i, j) with $i \ne j$. We see the coeffieients of N and $1/N$ in the exponent of N are both given by $-\frac{1}{12}\sum\limits_{i\ne j}\frac{\omega_i}{\omega_j}$. ∎

Lemma 2.5

$$\frac{\rho_3(\boldsymbol{\omega})}{\rho_3(N\omega_1, N\omega_2, \omega_3)} = N^{\frac{1}{24}\left(21-\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right)-N\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)-\frac{1}{N}\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)\right)} \prod_{k=1}^{N-1} \Gamma_3\left(\frac{k\omega_3}{N}; \boldsymbol{\omega}\right).$$

Proof. We again apply Lemma 2.2 to the case $r = 3$, $N_1 = N_2 = 1$, $N_3 = N$. Then

$$\frac{\rho_3(\omega_1, \omega_2, \omega_3)}{\rho_3(\omega_1, \omega_2, \frac{\omega_3}{N})} = \prod_{k=1}^{N-1} \Gamma_3\left(\frac{k\omega_3}{N}; \boldsymbol{\omega}\right).$$

By Lemma 2.1

$$\begin{aligned}\rho_3(\omega_1,\omega_2,\frac{\omega_3}{N}) &= \rho_3(N\omega_1,N\omega_2,\omega_3)N^{1-\frac{1}{24}\left(\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right)+N\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)+\frac{1}{N}\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)+3\right)}\\ &= \rho_3(N\omega_1,N\omega_2,\omega_3)N^{\frac{1}{24}\left(21-\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right)-N\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)-\frac{1}{N}\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)\right)}.\end{aligned}$$

Hence we have the conclusion. ∎

Lemma 2.6 *(a) For a sequence of positive numbers ω_j $(j=1,2,3)$, we have*

$$\prod_{l=1}^{3}\frac{\rho_3(\omega_i,\omega_j,\omega_l)}{\rho_3(N\omega_i,N\omega_j,\omega_l)} = N^{\frac{21}{8}-\frac{1}{24}\left(1+N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{k=1}^{N-1}\prod_{l=1}^{3}\Gamma_3\left(\frac{k\omega_l}{N};(\omega_1,\omega_2,\omega_3)\right).$$

(b) For a sequence of positive numbers ω_j $(j=1,2,3,4)$, we have

$$\prod_{l=1}^{4}\prod_{\substack{i,j\neq l\\ i<j}}\frac{\rho_3(\omega_i,\omega_j,\omega_l)}{\rho_3(N\omega_i,N\omega_j,\omega_l)} = N^{\frac{21}{2}-\frac{1}{12}\left(1+N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{k=1}^{N-1}\prod_{l=1}^{4}\prod_{\substack{i,j\neq l\\ i<j}}\Gamma_3\left(\frac{k\omega_l}{N};(\omega_i,\omega_j,\omega_l)\right)$$

Proof. We take a product of the result of the previous lemma over distinct $i,j,k\in\{1,2,3,4\}$ with $i<j$. The product consists of twelve terms. The coeffients of 1, N and $1/N$ in the exponent are all $\sum\limits_{i\neq j}\frac{\omega_i}{\omega_j}$. ∎

Lemma 2.7 *For any $j\in 1,2,...,r$, it holds that*

$$\Gamma_r(x+\omega_j;\boldsymbol{\omega}) = \Gamma_r(x;\boldsymbol{\omega})\Gamma_{r-1}(x;\boldsymbol{\omega}(j))^{-1}.$$

Proof. We prove for $j=1$. The general cases are similarly shown. By definition we have $\Gamma_r(x+\omega_1)=e^{\zeta_r'(0,x+\omega_1;\boldsymbol{\omega})}$ Here we compute

$$\begin{aligned}\zeta_r'(0,x+\omega_1;\boldsymbol{\omega}) &= \left(\sum_{k_1\geq 0}\sum_{k_2\geq 0}(k_1\omega_1+k_2\omega_2+\cdots+k_r\omega_r+x+\omega_1)^{-s}\right)'_{s=0}\\ &= \left(\sum_{k_1\geq 1}\sum_{k_2\geq 0}\cdots\sum_{k_r\geq 0}(k_1\omega_1+k_2\omega_2+\cdots+k_r\omega_r+x)^{-s}\right)'_{s=0}\\ &= \zeta_2'(0,x;\boldsymbol{\omega})-\left(\sum_{k_2\geq 0}\cdots\sum_{k_r\geq 0}(k_2\omega_2+\cdots+k_r\omega_r+x)^{-s}\right)'_{s=0}\\ &= \zeta_r'(0,x;\boldsymbol{\omega})-\zeta_{r-1}'(0,x;\boldsymbol{\omega}(1)).\end{aligned}$$

Hence $\Gamma_r(x+\omega_1;\boldsymbol{\omega}) = e^{\zeta_r'(0,x;\boldsymbol{\omega})-\zeta_{r-1}'(0,x;\boldsymbol{\omega}(1))} = \Gamma_r(x;\boldsymbol{\omega})\Gamma_{r-1}(x;\boldsymbol{\omega}(1))^{-1}$. ∎

Lemma 2.8

$$\frac{\rho_3(\omega_1,\omega_2,\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)} = \frac{\sqrt{\omega_1\omega_2\omega_3}}{(2\pi)^{\frac{3}{2}}\Gamma_3(|\boldsymbol{\omega}|;\boldsymbol{\omega})}.$$

Proof. We compute

$$\begin{aligned}
&\frac{\rho_3(\omega_1,\omega_2,\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)}\\
&= \lim_{x\to 0}\frac{x^3\Gamma_2(x;(\omega_1,\omega_2))\Gamma_2(x;(\omega_2,\omega_3))\Gamma_2(x;(\omega_1,\omega_3))}{x\Gamma_3(x;\boldsymbol{\omega})}\\
&= \lim_{x\to 0} x^2\exp\left(\zeta_2'(0,x;(\omega_1,\omega_2))+\zeta_2'(0,x;(\omega_2,\omega_3))+\zeta_2'(0,x;(\omega_1,\omega_3))-\zeta_3'(0,x;\boldsymbol{\omega})\right).
\end{aligned}$$

Here

$$\begin{aligned}
&\zeta_2(s,x;(\omega_1,\omega_2))+\zeta_2(s,x;(\omega_2,\omega_3))+\zeta_2(s,x;(\omega_1,\omega_3))-\zeta_3(s,x;\boldsymbol{\omega})\\
&= \sum_{\mathbf{m}\ge 0}\big((m_1\omega_1+m_2\omega_2+x)^{-s}+(m_2\omega_2+m_3\omega_3+x)^{-s}+(m_1\omega_1+m_3\omega_3+x)^{-s}\\
&\qquad -(m_1\omega_1+m_2\omega_2+m_3\omega_3+x)^{-s}\big)\\
&= -\sum_{\mathbf{m}\ge 1}(\mathbf{m}\cdot\boldsymbol{\omega}+x)^{-s}+\sum_{m_1=0}^{\infty}(m_1\omega_1+x)^{-s}+\sum_{m_2=0}^{\infty}(m_2\omega_2+x)^{-s}+\sum_{m_3=0}^{\infty}(m_3\omega_3+x)^{-s}\\
&= -\zeta_3(s,x+|\boldsymbol{\omega}|;\boldsymbol{\omega})+\sum_{j=1}^{3}\zeta_1(s,x,\omega_j)-x^{-s}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\rho_3(\omega_1,\omega_2,\omega_3)}{\rho_2(\omega_1,\omega_2)\rho_2(\omega_2,\omega_3)\rho_2(\omega_1,\omega_3)}\\
&= \lim_{x\to 0} x^2\exp\left(-\zeta_3'(0,x+|\boldsymbol{\omega}|;\boldsymbol{\omega})+\sum_{j=1}^{3}\zeta_1'(0,x,\omega_j)+\log x\right)\\
&= \lim_{x\to 0} x^3\exp\left(-\zeta_3'(0,x+|\boldsymbol{\omega}|;\boldsymbol{\omega})+\sum_{j=1}^{3}\zeta_1'(0,x,\omega_j)\right)\\
&= \left(\Gamma_3(|\boldsymbol{\omega}|;\boldsymbol{\omega})\prod_{j=1}^{3}\rho_1(\omega_j)\right)^{-1}.
\end{aligned}$$

As we have

$$\begin{aligned}
\rho_1(\omega_j)^{-1} &= \lim_{x\to 0} x\Gamma_1(x,\omega_j) \\
&= \lim_{x\to 0} x e^{\zeta_1'(0,x,\omega_j)} \\
&= \lim_{x\to 0} x e^{\left(\sum_{m=0}^{\infty}(m\omega_j+x)^{-s}\right)'_{s=0}} \\
&= \lim_{x\to 0} x e^{\left(x^{-s}+\sum_{m=1}^{\infty}(m\omega_j+x)^{-s}\right)'_{s=0}} \\
&= \left(\lim_{x\to 0} x e^{-\log x}\right) e^{(\omega_j^{-s}\zeta(s))'_{s=0}} \\
&= e^{(-\log\omega_j)\zeta(0)+\zeta'(0)} \\
&= e^{\frac{1}{2}\log\omega_j-\frac{1}{2}\log(2\pi)} \\
&= \sqrt{\frac{\omega_j}{2\pi}},
\end{aligned}$$

we reach the conclusion. ∎

Lemma 2.9

$$\frac{\rho_3(\omega_1,\omega_2,\omega_3)}{\rho_3(\omega_1,\omega_2,N\omega_3)} = N^{\frac{1}{24}\left(21-\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+\frac{1}{N}\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)+N\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)\right)\right)} \prod_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};(\omega_1,\omega_2,\omega_3)\right)$$

Proof. When $r=3$, $N_1=N_2=N$ and $N_3=1$, Lemma 2.2 shows that

$$\frac{\rho_3(\omega_1,\omega_2,\omega_3)}{\rho_3(\frac{\omega_1}{N},\frac{\omega_2}{N},\omega_3)} = \prod_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};(\omega_1,\omega_2,\omega_3)\right).$$

Here by Lemma 2.1

$$\begin{aligned}
\rho_3\left(\frac{\omega_1}{N},\frac{\omega_1}{N},\omega_3\right) &= \rho_3(\omega_1,\omega_2,N\omega_3)N^{1-\frac{1}{24}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+N\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)+\frac{1}{N}\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)\right)} \\
&= \rho_3(\omega_1,\omega_2,N\omega_3)N^{\frac{1}{24}\left(21-\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-N\left(\frac{\omega_1}{\omega_3}+\frac{\omega_2}{\omega_3}\right)-\frac{1}{N}\left(\frac{\omega_3}{\omega_1}+\frac{\omega_3}{\omega_2}\right)\right)}.
\end{aligned}$$

∎

Lemma 2.10 *For a sequence of positive numbers ω_j $(j=1,2,3)$, we have*

$$\prod_{l=1}^{3}\frac{\rho_3(\omega_i,\omega_j,\omega_l)}{\rho_3(\omega_i,\omega_j,N\omega_l)} = N^{\frac{21}{8}-\frac{1}{24}\left(1+N+\frac{1}{N}\right)\sum_{i\ne j}\frac{\omega_i}{\omega_j}} \prod_{i<j}\prod_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right).$$

For a sequence of positive numbers ω_j $(j=1,2,3,4)$, we have

$$\prod_{l=1}^{4}\prod_{\substack{i,j\neq l\\ i<j}}\frac{\rho_3(\omega_i,\omega_j,\omega_l)}{\rho_3(\omega_i,\omega_j,N\omega_l)} = N^{\frac{21}{2}-\frac{1}{12}\left(1+N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{l=1}^{4}\prod_{\substack{i,j\neq l\\ i<j}}\prod_{0\le k_1,k_2\le N-1}{}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_l)\right).$$

Proof. By the previous lemma, we have an expression for $\frac{\rho_3(\omega_i,\omega_j,\omega_l)}{\rho_3(\omega_i,\omega_j,N\omega_l)}$. Taking a product over $l=1,2,3$ leads to the conclusion for the first identity.

For the second identity, all we have to do is to take a product over possible combinations i,j,l where $l=1,2,3,4$ and (i,j) runs through combinations of elements not equal to l. ∎

3 The case $r=3$

Theorem 3.1 *In the product* (1.2) *for $r=3$, the contribution from terms with $k_2=k_3=0$ is expressed as*

$$\begin{aligned}&\prod_{k=1}^{N-1}S_3\left(\frac{k\omega_1}{N};\ \boldsymbol{\omega}\right)\\ =\ &\exp\Bigg(-2\zeta_3'(0;\ (\omega_1,N\omega_2,N\omega_3))+2\zeta_3'(0;\ \boldsymbol{\omega})\\ &-\zeta_2'(0;\ (\omega_1,N\omega_2))-\zeta_2'(0;\ (\omega_1,N\omega_3))+\zeta_2'(0;\ (\omega_1,\omega_2))+\zeta_2'(0;\ (\omega_1,\omega_3))\\ &-(\log N)\Big(2\zeta_3(0;\ (\omega_1,N\omega_2,N\omega_3))+\zeta_2(0;\ (\omega_1,N\omega_2))+\zeta_2(0;\ (\omega_1,N\omega_3))+\zeta(0)\Big)\Bigg).\end{aligned}$$

The total contribution from terms with two of the coefficients k_j being zero is expressed as follows:

$$\prod_{k=1}^{N-1}S_3\left(\frac{k\omega_1}{N};\ \boldsymbol{\omega}\right)S_3\left(\frac{k\omega_2}{N};\ \boldsymbol{\omega}\right)S_3\left(\frac{k\omega_3}{N};\ \boldsymbol{\omega}\right)=N^{\frac{3}{2}}\prod_{k=1}^{N-1}\frac{\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)}{\prod_{j=1}^{3}\Gamma_3\left(\frac{k\omega_j}{N}\right)^2}.$$

Proof.

$$\begin{aligned}&\prod_{k_1=1}^{N-1}S_3\left(\frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right)\\ =\ &\prod_{k_1=1}^{N-1}\frac{1}{\Gamma_3\left(|\boldsymbol{\omega}|-\frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right)\Gamma_3\left(\frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right)}\\ =\ &\exp\left(-\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{k_1=1}^{N-1}\left(\zeta_3\left(s,\ |\boldsymbol{\omega}|-\frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right)+\zeta_3\left(s,\ \frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right)\right)\right).\end{aligned}$$

Here

$$
\begin{aligned}
&\sum_{k_1=1}^{N-1} \zeta_3\left(s,\ |\boldsymbol{\omega}| - \frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right) \\
&\quad = \sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + |\boldsymbol{\omega}| - \frac{k_1\omega_1}{N}\right)^{-s} \\
&\quad = \sum_{k_1=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\left(m_1\omega_1 + m_2\omega_2 + m_3\omega_3 - \frac{k_1\omega_1}{N}\right)^{-s} \\
&\quad = N^s\sum_{k_1=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}((m_1N - k_1)\omega_1 + m_2N\omega_2 + m_3N\omega_3)^{-s} \\
&\quad = N^s\left(\zeta_3(s; (\omega_1,\ N\omega_2,\ N\omega_3)) - \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}(m_1N\omega_1 + m_2N\omega_2 + m_3N\omega_3)^{-s}\right) \\
&\quad = N^s\left(\zeta_3(s;\ (\omega_1,\ N\omega_2,\ N\omega_3)) - N^{-s}\zeta_3(s;\ \boldsymbol{\omega})\right) \\
&\quad = N^s\zeta_3(s;\ (\omega_1,\ N\omega_2,\ N\omega_3)) - \zeta_3(s;\ \boldsymbol{\omega}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k_1=1}^{N-1} \zeta_3\left(s,\ \frac{k_1\omega_1}{N};\ \boldsymbol{\omega}\right) \\
&\quad = \sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + \frac{k_1\omega_1}{N}\right)^{-s} \\
&\quad = N^s\sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}((m_1N + k_1)\omega_1 + m_2N\omega_2 + m_3N\omega_3)^{-s} \\
&\quad = N^s(\zeta_3(s;\ (\omega_1,\ N\omega_2,\ N\omega_3)) - N^{-s}\zeta_3(s;\ \boldsymbol{\omega})) \\
&\qquad + N^s\sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=1}^{\infty}((m_1N + k_1)\omega_1 + m_2N\omega_2)^{-s} \\
&\qquad + N^s\sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_3=1}^{\infty}((m_1N + k_1)\omega_1 + m_3N\omega_3)^{-s} \\
&\qquad + N^s\sum_{k_1=1}^{N-1}\sum_{m_1=0}^{\infty}((m_1N + k_1)\omega_1)^{-s} \\
&\quad = N^s\zeta_3(s;\ (\omega_1,\ N\omega_2,\ N\omega_3)) - \zeta_3(s;\ \boldsymbol{\omega}) \\
&\qquad + N^s(\zeta_2(s;\ (\omega_1, N\omega_2)) - N^{-s}\zeta_2(s;\ (\omega_1,\omega_2)))
\end{aligned}
$$

$$
\begin{aligned}
&\quad +N^s(\zeta_2(s;\ (\omega_1, N\omega_3)) - N^{-s}\zeta_2(s;\ (\omega_1,\omega_3)))\\
&\quad +N^s\omega_1^{-s}(1-N^{-s})\zeta(s)\\
&= N^s\big(\zeta_3(s;\ (\omega_1, N\omega_2, N\omega_3)) + \zeta_2(s;\ (\omega_1, N\omega_2)) + \zeta_2(s;\ (\omega_1, N\omega_3)) + \omega_1^{-s}\zeta(s)\big)\\
&\quad -\zeta_3(s;\ \boldsymbol{\omega}) - \zeta_2(s;\ (\omega_1,\omega_2)) - \zeta_2(s;\ (\omega_1,\omega_3)) - \omega_1^{-s}\zeta(s).
\end{aligned}
$$

Thus we obtain the first identity in Theorem. Therefore we calculate

$$
\begin{aligned}
&\log \prod_{k=1}^{N-1} S_3\left(\frac{k\omega_1}{N};\ \boldsymbol{\omega}\right) S_3\left(\frac{k\omega_2}{N};\ \boldsymbol{\omega}\right) S_3\left(\frac{k\omega_2}{N};\ \boldsymbol{\omega}\right)\\
&= -2\sum_{l=1}^{3}\big(\zeta_3'(0;\boldsymbol{\omega}_l^N) - \zeta_3'(0;\boldsymbol{\omega})\big) - \sum_{i\neq j}\big(\zeta_2'(0;(\omega_i, N\omega_j)) - \zeta_2'(0;(\omega_i,\omega_j))\big)\\
&\quad -(\log N)\left(2\sum_{l=1}^{3}\zeta_3(0;\boldsymbol{\omega}_l^N) + \sum_{i,j}\zeta_2(0;(\omega_i, N\omega_j)) - \frac{3}{2}\right)\\
&= \sum_{l=1}^{3}\log\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)}{\rho_3(\boldsymbol{\omega})}\frac{\rho_2(\omega_l,\omega_i)\rho_2(\omega_l,\omega_j)\rho_2(\omega_i,\omega_j)}{N\rho_2(\omega_l, N\omega_i)\rho_2(\omega_l, N\omega_j)\rho_2(N\omega_i, N\omega_j)}\right)^2 + \sum_{i\neq j}\log\frac{\rho_2(\omega_i, N\omega_j)\sqrt{N}}{\rho_2(\omega_i,\omega_j)}\\
&\quad -(\log N)\left(-\frac{1}{12}\left(1+N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j} - \frac{9}{12} + \frac{1}{12}\left(N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j} + \frac{18}{12} - \frac{3}{2}\right)\\
&= \sum_{l=1}^{3}\log\left(\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)}{\rho_3(\boldsymbol{\omega})}\right)^2\frac{\rho_2(\omega_l,\omega_i)\rho_2(\omega_l,\omega_j)}{\rho_2(\omega_l, N\omega_i)\rho_2(\omega_l, N\omega_j)}\right) + \log N^{\frac{3}{2}-\frac{1}{6}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\\
&\quad +(\log N)\left(\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j} + \frac{9}{12}\right)\\
&= \log\left(N^{\frac{9}{4}-\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{l=1}^{3}\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)}{\rho_3(\boldsymbol{\omega})}\right)^2\frac{\rho_2(\omega_l,\omega_i)\rho_2(\omega_l,\omega_j)}{\rho_2(\omega_l, N\omega_i)\rho_2(\omega_l, N\omega_j)}\right),
\end{aligned}
$$

where i and j denote the elements not being equal to l, when they are in a sum or a product over l.

Hence by Lemmas in the previous section,

$$
\begin{aligned}
&\prod_{k=1}^{N-1} S_3\left(\frac{k\omega_1}{N};\ \boldsymbol{\omega}\right) S_3\left(\frac{k\omega_2}{N};\ \boldsymbol{\omega}\right) S_3\left(\frac{k\omega_2}{N};\ \boldsymbol{\omega}\right)\\
&= N^{\frac{9}{4}-\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{l=1}^{3}\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)}{\rho_3(\boldsymbol{\omega})}\right)^2\prod_{\substack{1\leq i,j\leq 3\\ i\neq j}}\frac{\rho_2(\omega_i,\omega_j)}{\rho_2(\omega_i, N\omega_j)}
\end{aligned}
$$

$$= N^{\left(\frac{9}{4}-\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}\right)+\left(-\frac{21}{4}+\frac{1}{12}\left(1+N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}\right)+\left(\frac{9}{2}-\frac{1}{12}\left(N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}\right)} \frac{\prod_{k=1}^{N-1}\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)}{\prod_{k=1}^{N-1}\prod_{j=1}^{3}\Gamma_3\left(\frac{k\omega_j}{N}\right)^2}$$

$$= N^{\frac{3}{2}}\frac{\prod_{k=1}^{N-1}\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)}{\prod_{k=1}^{N-1}\prod_{j=1}^{3}\Gamma_3\left(\frac{k\omega_j}{N}\right)^2}.$$

∎

Theorem 3.2 *In the product* (1.2) *for $r=3$, the contribution from terms with $k_3=0$ is expressed as*

$$\begin{aligned}\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}S_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right) &= \exp\Bigg(-2\left(\zeta_3'(0;\boldsymbol{\omega}_3^{(N)})-\zeta_3'(0;\boldsymbol{\omega}_1^N)-\zeta_3'(0;\boldsymbol{\omega}_2^N)+\zeta_3'(0;\boldsymbol{\omega})\right)\\ &- \left(2\zeta_2'(0;(\omega_1,\omega_2))-\zeta_2'(0;(N\omega_1,\omega_2))-\zeta_2'(0;(\omega_1,N\omega_2))\right)\\ &- (\log N)\Bigg(2\left(\zeta_3(0;\boldsymbol{\omega}_3^{(N)})-\zeta_3(0;\boldsymbol{\omega}_1^N)-\zeta_3(0;\boldsymbol{\omega}_2^N)\right)\\ &+ \left(\zeta_2(0;(\omega_1,\omega_2))-\zeta_2(0;(N\omega_1,\omega_2))-\zeta_2(0;(\omega_1,N\omega_2))\right)\Bigg)\Bigg)\end{aligned}$$

The total contribution from terms with only one of the coefficients k_j being zero is expressed as follows:

$$\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}S_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)S_3\left(\frac{k_1\omega_2+k_2\omega_3}{N};\ \boldsymbol{\omega}\right)S_3\left(\frac{k_1\omega_3+k_2\omega_1}{N};\ \boldsymbol{\omega}\right)$$
$$= \frac{N^{-\frac{9}{4}+\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{j=1}^{3}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N};\boldsymbol{\omega}\right)^4}{\left(\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)\left(\prod_{i<j}\prod_{0\le k_1,k_2\le N-1}{}'\ \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)^2\right)}.$$

Proof.

$$
\begin{aligned}
&\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1} S_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)\\
&= \prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1} \frac{1}{\Gamma_3\left(|\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)\Gamma_3\left(\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)}\\
&= \exp\left(-\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\left(\zeta_3\left(s,\ |\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)+\zeta_3\left(s,\ \frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)\right)\right).
\end{aligned}
$$

Here

$$
\begin{aligned}
&\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\zeta_3\left(s,\ |\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)\\
&= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3+|\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2}{N}\right)^{-s}\\
&= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3-\frac{k_1\omega_1+k_2\omega_2}{N}\right)^{-s}\\
&= N^s\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\left((m_1N-k_1)\omega_1+(m_2N-k_2)\omega_2+m_3N\omega_3\right)^{-s}\\
&= N^s\sum_{\substack{m_1\geq 1\\ N\nmid m_1}}\sum_{\substack{m_2\geq 1\\ N\nmid m_2}}\sum_{m_3\geq 1}\left(m_1\omega_1+m_2\omega_2+m_3N\omega_3\right)^{-s}\\
&= N^s\left(\zeta_3(s;(\omega_1,\omega_2,N\omega_3))-\zeta_3(s;(N\omega_1,\omega_2,N\omega_3))-\zeta_3(s;(\omega_1,N\omega_2,N\omega_3))+\zeta_3(s,N(\omega_1,\omega_2,\omega_3))\right)\\
&= N^s\left(\zeta_3(s;\boldsymbol{\omega}_3^{(N)})-\zeta_3(s;\boldsymbol{\omega}_1^{N})-\zeta_3(s;\boldsymbol{\omega}_2^{N})+\zeta_3(s,N\boldsymbol{\omega})\right)\\
&= N^s\left(\zeta_3(s;\boldsymbol{\omega}_3^{(N)})-\zeta_3(s;\boldsymbol{\omega}_1^{N})-\zeta_3(s;\boldsymbol{\omega}_2^{N})\right)+\zeta_3(s;\boldsymbol{\omega}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\zeta_3\left(s,\ \frac{k_1\omega_1+k_2\omega_2}{N};\ \boldsymbol{\omega}\right)\\
&= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3+\frac{k_1\omega_1+k_2\omega_2}{N}\right)^{-s}\\
&= N^s\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left((m_1N+k_1)\omega_1+(m_2N+k_2)\omega_2+m_3N\omega_3\right)^{-s}
\end{aligned}
$$

$$= N^s \left(\sum_{\substack{m_1\ge 1\\ N\nmid m_1}} \sum_{\substack{m_2\ge 1\\ N\nmid m_2}} \sum_{m_3\ge 1} (m_1\omega_1+m_2\omega_2+m_3N\omega_3)^{-s} + \sum_{\substack{m_1\ge 1\\ N\nmid m_1}} \sum_{\substack{m_2\ge 1\\ N\nmid m_2}} (m_1\omega_1+m_2\omega_2)^{-s} \right)$$

$$= N^s \Big(\zeta_3(s;\boldsymbol{\omega}_3^{(N)}) - \zeta_3(s;\boldsymbol{\omega}_1^N) - \zeta_3(s;\boldsymbol{\omega}_2^N) + \zeta_3(s,N\boldsymbol{\omega})$$

$$+\zeta_2(s;(\omega_1,\omega_2) - \zeta_2(s;(N\omega_1,\omega_2) - \zeta_2(s;(\omega_1,N\omega_2)) + \zeta_2(s;(N\omega_1,N\omega_2)\Big)$$

$$= N^s \Big(\zeta_3(s;\boldsymbol{\omega}_3^{(N)}) - \zeta_3(s;\boldsymbol{\omega}_1^N) - \zeta_3(s;\boldsymbol{\omega}_2^N)\Big) + \zeta_3(s;\boldsymbol{\omega})$$

$$+N^s\Big(\zeta_2(s;(\omega_1,\omega_2)) - \zeta_2(s;(N\omega_1,\omega_2)) - \zeta_2(s;(\omega_1,N\omega_2))\Big) + \zeta_2(s;(\omega_1,\omega_2))$$

Thus we have the first identity in Theorem. For the second identity we calculate

$$\frac{\partial}{\partial s}\Big|_{s=0} \sum_{i<j} \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{N-1} \left(\zeta_3\left(s,\ |\boldsymbol{\omega}| - \frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right) + \zeta_3\left(s,\ \frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)\right)$$

$$= 2\sum_{l=1}^{3} \Big(\zeta_3'(0;\boldsymbol{\omega}_l^{(N)}) - 2\zeta_3'(0;\boldsymbol{\omega}_l^N) + \zeta_3'(0;\boldsymbol{\omega})\Big) + \sum_{i\neq j}\Big(\zeta_2'(0;(\omega_i,\omega_j)) - \zeta_2'(0;(N\omega_i,\omega_j))\Big)$$

$$+ (\log N)\left(2\sum_{l=1}^{3}\Big(\zeta_3(0;\boldsymbol{\omega}_l^{(N)}) - 2\zeta_3(0;\boldsymbol{\omega}_l^N)\Big) + \sum_{i\neq j}\Big(\frac{\zeta_2(0;(\omega_i,\omega_j))}{2} - \zeta_2(0;(N\omega_i,\omega_j))\Big)\right)$$

$$= 2\log\left(\prod_{i\neq j}\frac{\rho_2(\omega_i,\omega_j)}{\rho_2(\omega_i,N\omega_j)}\right)\left(N^{3-\frac{1}{12}\left(\sum_{i\neq j}\frac{\omega_i}{\omega_j}+9\right)}\right)^2 \prod_{l=1}^{3}\left(\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)^2}{\rho_3(\boldsymbol{\omega})}\right)^2 \frac{\rho_3(\boldsymbol{\omega})}{\rho_3(\boldsymbol{\omega}_l^{(N)})}\frac{\sqrt{N}}{\left(\sqrt{N^2}\right)^2}\right)$$

$$+\log\prod_{i\neq j}\left(\frac{\rho_2(N\omega_i,\omega_j)}{\rho_2(\omega_i,\omega_j)}\sqrt{N}\right)$$

$$+(\log N)\left(\frac{1+N+\frac{1}{N}}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j} + \frac{9}{12} + \frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j} + \frac{9}{12} - \frac{N+\frac{1}{N}}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j} - \frac{18}{12}\right)$$

$$= \log\left(\prod_{i\neq j}\frac{\rho_2(\omega_i,\omega_j)}{\rho_2(\omega_i,N\omega_j)}\right)\left(N^{3-\frac{1}{3}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\right)\prod_{l=1}^{3}\left(\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)^2}{\rho_3(\boldsymbol{\omega})}\right)^2\frac{\rho_3(\boldsymbol{\omega})}{\rho_3(\boldsymbol{\omega}_l^{(N)})}\right)^2$$

$$+(\log N)\left(\frac{1}{6}\sum_{i\neq j}\frac{\omega_i}{\omega_j}\right)$$

$$= \log\left(\prod_{i\neq j}\frac{\rho_2(\omega_i,\omega_j)}{\rho_2(\omega_i,N\omega_j)}\right)\left(N^{3-\frac{1}{6}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\right)\prod_{l=1}^{3}\left(\left(\frac{\rho_3(\boldsymbol{\omega}_l^N)^2}{\rho_3(\boldsymbol{\omega})}\right)^2\frac{\rho_3(\boldsymbol{\omega})}{\rho_3(\boldsymbol{\omega}_l^{(N)})}\right)^2$$

$$= \log\left(N^{3-\frac{1}{6}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\right)\left(N^{\frac{9}{2}-\frac{1}{12}(N+\frac{1}{N})\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)$$

$$+\log\left(N^{-\frac{21}{8}+\frac{1}{24}(1+N+\frac{1}{N})\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\frac{\prod_{i<j}\prod'_{0\le k_1,k_2\le N-1}\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N}\right)}{\prod_{j=1}^{3}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N}\right)^2}\right)^2$$

$$= \log\left(N^{\frac{9}{4}-\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{i\neq j}^{3}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\left(\frac{\prod_{i<j}\prod'_{0\le k_1,k_2\le N-1}\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N}\right)}{\prod_{j=1}^{3}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N}\right)^2}\right)^2\right).$$

∎

Example. When $N=2$, $\boldsymbol{\omega}=(1,1,1)$, we have

$$S_3(1)^3 = 2^{-\frac{7}{4}}\left(\frac{\Gamma_3(\frac{1}{2})^6}{\Gamma_2(\frac{1}{2})^3\Gamma_3(\frac{1}{2})^6\Gamma_3(1)^3}\right)^2 = 2^{-\frac{7}{4}}\left(\Gamma_2\left(\frac{1}{2}\right)\Gamma_3(1)\right)^{-6}.$$

As we will show later in the final section, it agrees to the results on the special values $S_3(1)=e^{-\zeta'(-2)}$, $\Gamma_2(\frac{1}{2})=2^{-\frac{7}{24}}e^{-\frac{\zeta'(-1)}{2}}$, and $\Gamma_3(1)=e^{\frac{\zeta'(-2)+\zeta'(-1)}{2}}$.

Theorem 3.3 *In the product* (1.2) *for* $r=3$*, the contribution from terms with none of the coefficients* k_j *being zero is expressed as follows:*

$$\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}\prod_{k_3=1}^{N-1}S_3\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)$$

$$= \exp\left(2\left(\sum_{j=1}^{3}\left(\zeta_3'(0;\boldsymbol{\omega}_j^{(N)})-\zeta_3'(0;\boldsymbol{\omega}_j^{N})\right)-(\log N)\zeta_3(0;\boldsymbol{\omega})\right)\right)$$

$$= N^{\frac{7}{4}-\frac{1}{12}\sum_{i<j}\frac{\omega_i}{\omega_j}}\left(\frac{\prod'_{0\le k_1,k_2\le N-1}\prod_{\substack{i<j\\ 1\le i,j\le 3}}\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)}{\prod_{k=1}^{N-1}\prod_{l=1}^{3}\Gamma_3\left(\frac{k\omega_l}{N};\boldsymbol{\omega}\right)}\right)^2.$$

Proof.

$$\begin{aligned}
&\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}\prod_{k_3=1}^{N-1} S_3\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\\
&\quad= \prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}\prod_{k_3=1}^{N-1} \frac{1}{\Gamma_3\left(|\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\Gamma_3\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)}\\
&\quad= \exp\left(-\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\left(\zeta_3\left(s,\ |\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\right.\right.\\
&\qquad\left.\left.+\zeta_3\left(s,\ \frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\right)\right).
\end{aligned}$$

Here

$$\begin{aligned}
&\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\zeta_3\left(s,\ |\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\\
&\quad= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3+|\boldsymbol{\omega}|-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N}\right)^{-s}\\
&\quad= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N}\right)^{-s}\\
&\quad= N^s\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}\left((m_1N-k_1)\omega_1+(m_2N-k_2)\omega_2+(m_3N-k_3)\omega_3\right)^{-s}\\
&\quad= N^s\sum_{\substack{m_1\geq 1\\ N\nmid m_1}}\sum_{\substack{m_2\geq 1\\ N\nmid m_2}}\sum_{\substack{m_3\geq 1\\ N\nmid m_3}}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3\right)^{-s}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\zeta_3\left(s,\ \frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)\\
&\quad= \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N}\right)^{-s}\\
&\quad= N^s\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\left((m_1N+k_1)\omega_1+(m_2N+k_2)\omega_2+(m_3N+k_3)\omega_3\right)^{-s}\\
&\quad= N^s\sum_{\substack{m_1\geq 1\\ N\nmid m_1}}\sum_{\substack{m_2\geq 1\\ N\nmid m_2}}\sum_{\substack{m_3\geq 1\\ N\nmid m_3}}\left(m_1\omega_1+m_2\omega_2+m_3\omega_3\right)^{-s}
\end{aligned}$$

Hence

$$
\begin{aligned}
&\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\sum_{k_3=1}^{N-1}\left(\zeta_3\left(s,\ |\boldsymbol{\omega}| - \frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\boldsymbol{\omega}\right)+\zeta_3\left(s,\ \frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\boldsymbol{\omega}\right)\right)\\
&= \frac{\partial}{\partial s}\bigg|_{s=0} 2N^s \sum_{\substack{m_1\geq 1\\ N\nmid m_1}}\sum_{\substack{m_2\geq 1\\ N\nmid m_2}}\sum_{\substack{m_3\geq 1\\ N\nmid m_3}} (m_1\omega_1+m_2\omega_2+m_3\omega_3)^{-s}\\
&= \frac{\partial}{\partial s}\bigg|_{s=0} 2N^s\left((1-N^{-s})\zeta_3(s;\boldsymbol{\omega}) - \sum_{j=1}^{3}\left(\zeta_3(s;\boldsymbol{\omega}_j^{(N)}) - \zeta_3(s;\boldsymbol{\omega}_j^{N})\right)\right)\\
&= \frac{\partial}{\partial s}\bigg|_{s=0} 2\left((N^s-1)\zeta_3(s;\boldsymbol{\omega}) - N^s\sum_{j=1}^{3}\left(\zeta_3(s;\boldsymbol{\omega}_j^{(N)}) - \zeta_3(s;\boldsymbol{\omega}_j^{N})\right)\right)\\
&= 2\left(-\sum_{j=1}^{3}\left(\zeta_3'(0;\boldsymbol{\omega}_j^{(N)}) - \zeta_3'(0;\boldsymbol{\omega}_j^{N})\right) + (\log N)\left(\zeta_3(0;\boldsymbol{\omega}) - \sum_{j=1}^{3}\left(\zeta_3(0;\boldsymbol{\omega}_j^{(N)}) - \zeta_3(0;\boldsymbol{\omega}_j^{N})\right)\right)\right)\\
&= 2\left(-\sum_{j=1}^{3}\left(\zeta_3'(0;\boldsymbol{\omega}_j^{(N)}) - \zeta_3'(0;\boldsymbol{\omega}_j^{N})\right) + (\log N)\zeta_3(0;\boldsymbol{\omega})\right),
\end{aligned}
$$

since

$$
\sum_{j=1}^{3}\zeta_3'(0;\boldsymbol{\omega}_j^{(N)}) = \sum_{j=1}^{3}\zeta_3'(0;\boldsymbol{\omega}_j^{N}) = \frac{1}{24}\left(\sum_{i\neq j}\frac{\omega_i}{\omega_j}\left(1+N+\frac{1}{N}\right)+9\right).
$$

Thus we have the first identity. For the second identity we compute

$$
\begin{aligned}
2\left(\zeta_3'(0;\boldsymbol{\omega}_j^{(N)}) - \zeta_3'(0;\boldsymbol{\omega}_j^{N})\right) &= \log\left(\frac{1}{N^{\frac{3}{2}}}\prod_{i<j}\frac{\rho_2(\omega_i,\omega_j)}{\rho_2(N\omega_i,N\omega_j)}\prod_{j=1}^{3}\frac{\rho_3(\boldsymbol{\omega}_j^{N})}{\rho_3(\boldsymbol{\omega}_j^{(N)})}\right)^2\\
&= \log N^{-3}\left(N^{3-\frac{1}{12}\left(\sum_{i\neq j}\frac{\omega_i}{\omega_j}+9\right)}\right)^2\left(\prod_{j=1}^{3}\frac{\rho_3(\boldsymbol{\omega}_j^{N})}{\rho_3(\boldsymbol{\omega}_j^{(N)})}\right)^2\\
&= \log N^{\frac{3}{2}-\frac{1}{6}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\left(\prod_{j=1}^{3}\frac{\rho_3(\boldsymbol{\omega}_j^{N})}{\rho_3(\boldsymbol{\omega}_j^{(N)})}\right)^2.
\end{aligned}
$$

We also compute by Lemmas in the previous section that

$$\begin{aligned}\frac{\rho_3(\boldsymbol{\omega}_j^N)}{\rho_3(\boldsymbol{\omega}_j^{(N)})} &= \frac{\rho_3(\boldsymbol{\omega}_j^N)}{\rho_3(\boldsymbol{\omega})}\frac{\rho_3(\boldsymbol{\omega})}{\rho_3(\boldsymbol{\omega}_j^{(N)})} \\ &= \frac{\prod\limits_{i<j}\prod\limits_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)}{\prod\limits_{l=1}^{3}\prod\limits_{k=0}^{N-1}\Gamma_3\left(\frac{k\omega_l}{N};\boldsymbol{\omega}\right)}.\end{aligned}$$

Thus

$$\begin{aligned}&\prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1}\prod_{k_3=1}^{N-1} S_3\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right) \\ &= N^{\frac{1}{12}\left(\sum\limits_{i\neq j}\frac{\omega_i}{\omega_j}+3\right)+\frac{3}{2}-\frac{1}{6}\sum\limits_{i\neq j}\frac{\omega_i}{\omega_j}}\left(\frac{\prod\limits_{i<j}\prod\limits_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)}{\prod\limits_{l=1}^{3}\prod\limits_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_l}{N};\boldsymbol{\omega}\right)}\right)^2 \\ &= N^{\frac{7}{4}-\frac{1}{12}\sum\limits_{i\neq j}\frac{\omega_i}{\omega_j}}\left(\frac{\prod\limits_{i<j}\prod\limits_{0\le k_1,k_2\le N-1}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)}{\prod\limits_{l=1}^{3}\prod\limits_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_l}{N};\boldsymbol{\omega}\right)}\right)^2.\end{aligned}$$

∎

Example. When $N=2$, $\boldsymbol{\omega}=(1,1,1)$, we have

$$S_3\left(\frac{3}{2}\right) = 2^{\frac{5}{4}}\left(\frac{\Gamma_3(\frac{1}{2})^6\Gamma_3(1)^3}{\Gamma_3(\frac{1}{2})^3}\right)^2 = 2^{\frac{5}{4}}\Gamma_3\left(\frac{1}{2}\right)^6\Gamma_3(1)^6.$$

This agrees to the results we will show in the final section, since $S_3\left(\frac{3}{2}\right) = 2^{-\frac{1}{8}}e^{\frac{3}{4}\zeta'(-2)}$.

4 The case $r=4$

Let

$$A = \prod_{1\le k_1,\dots,k_4\le N-1} S_4\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right).$$

Then

$$\prod_{0\le k_1,\dots,k_4\le N-1}' S_4\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) = A\prod_{j=1}^{4}A_j\prod_{\substack{j,l=1\\ j<l}}^{4}A_{j,l}\prod_{\substack{j,l,m=1\\ j<l<m}}^{4}A_{j,l,m}, \quad (4.1)$$

where A_j, $A_{j,l}$, and $A_{j,l,m}$ are the partial products defined in Section 1.

Theorem 4.1 *The partial product $A_{j,l,m}$ is expressed in terms of the triple sine functions as*

$$A_{j,l,m} = N^{-\frac{1}{4}} \prod_{k_n=1}^{N-1} \left(S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right)\right)^{\frac{1}{2}}. \qquad (4.2)$$

In the product (1.2) for $r = 4$, the total contribution from terms with three of the coefficients k_j being zero is

$$\prod_{j<l<m} A_{j,l,m} = N^2 \prod_{j=1}^{4} \prod_{k=1}^{N-1} \frac{\prod_{i\neq l} \Gamma_2\left(\frac{k\omega_l}{N}; (\omega_i, \omega_l)\right)^{\frac{1}{2}}}{\prod_{\substack{1\leq l\leq 4\\ l\neq j}} \Gamma_3\left(\frac{k\omega_l}{N}; \boldsymbol{\omega}(j)\right)^{\frac{1}{2}}}.$$

Proof. We calculate $A_{j,l,m}$ for $j,l,m = 1,2,3,4$ with $j < l < m$. Let $n \in \{1,2,3,4\}$ be the number which is neither j, l nor m. Then we have

$$\begin{aligned}
A_{j,l,m}^2 &= \prod_{k_n=1}^{N-1} S_4\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) S_4\left(\frac{(N-k_n)\omega_n}{N};\ \boldsymbol{\omega}\right) \\
&= \prod_{k_n=1}^{N-1} S_4\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_n - \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) \\
&= \prod_{k_n=1}^{N-1} S_4\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_j + \omega_l + \omega_m + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right)^{-1} \\
&= \prod_{k_n=1}^{N-1} S_4\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_j + \omega_l + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right)^{-1} S_3\left(\omega_j + \omega_l + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right) \\
&= \prod_{k_n=1}^{N-1} S_4\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_j + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}\right)^{-1} \\
&\qquad S_3\left(\omega_j + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_3\left(\omega_j + \omega_l + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right) \\
&= \prod_{k_n=1}^{N-1} S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\omega_j + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_3\left(\omega_j + \omega_l + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right). \\
&= \prod_{k_n=1}^{N-1} S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l,j)\right)^{-1} \\
&\qquad S_3\left(\omega_j + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right) S_2\left(\omega_j + \frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,l)\right)^{-1}.
\end{aligned}$$

$$= \prod_{k_n=1}^{N-1} S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l,j)\right)^{-1}$$
$$S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right) S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,j)\right)^{-1} S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,l)\right)^{-1} S_1\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,l,j)\right).$$

Hence we obtain

$$\begin{aligned} A_{j,l,m} &= \prod_{k_n=1}^{N-1} \left(S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right)\right)^{\frac{1}{2}} \\ &\quad \left(S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l,j)\right) S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,j)\right) S_2\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,l)\right)\right)^{-\frac{1}{2}} \\ &\quad S_1\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m,l,j)\right)^{\frac{1}{2}} \\ &= N^{-\frac{1}{4}} \prod_{k_n=1}^{N-1} \left(S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right) S_3\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(m)\right)\right)^{\frac{1}{2}} \end{aligned}$$

by (1.2) for $r = 1$ and 2. This proves (4.2). Theorem 3.1 leads to the second identity. ∎

Theorem 4.2 *The partial product $A_{i,j}$ is expressed in terms of the triple sine functions as*

$$A_{j,l} = \prod_{1\le k_m,k_n\le N-1} S_3\left(\frac{k_m\omega_m + k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right)^{\frac{1}{2}} S_3\left(\frac{k_m\omega_m + k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right)^{\frac{1}{2}}. \quad (4.3)$$

In the product (1.2) for $r = 4$, the total contribution from terms with exactly two of the coefficients k_j being zero is

$$\prod_{\substack{1\le i,j\le 4\\ i\ne j}} A_{i,j} = N^{-\frac{9}{2}+\frac{1}{12}\sum\limits_{j\ne l}\frac{\omega_j}{\omega_l}} \frac{\prod\limits_{l=1}^{4}\prod\limits_{j\ne l}\prod\limits_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N};\boldsymbol{\omega}(l)\right)^2}{\prod\limits_{l=1}^{4}\left(\prod\limits_{\substack{i<j\\ i,j\ne l}}\ \prod\nolimits'_{0\le k_1,k_2\le N-1}\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}(l)\right)\right)\left(\prod\limits_{i,j\ne l}\prod\limits_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)}.$$

Proof. We first calculate $A_{1,2}$. It follows that

$$\begin{aligned}
A_{1,2}^2 &= \prod_{1\le k_3,k_4\le N-1} S_4\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right)^2 \\
&= \prod_{1\le k_3,k_4\le N-1} S_4\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) S_4\left(\frac{(N-k_3)\omega_3+(N-k_4)\omega_4}{N};\ \boldsymbol{\omega}\right) \\
&= \prod_{1\le k_3,k_4\le N-1} S_4\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_3+\omega_4-\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) \\
&= \prod_{1\le k_3,k_4\le N-1} S_4\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_1+\omega_2+\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right)^{-1} \\
&= \prod_{1\le k_3,k_4\le N-1} S_4\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_2+\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}\right)^{-1} \\
&\qquad\qquad \times S_3\left(\omega_2+\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}(1)\right) \\
&= \prod_{1\le k_3,k_4\le N-1} S_3\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}(2)\right) S_3\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}(1)\right) \\
&\qquad\qquad \times S_2\left(\frac{k_3\omega_3+k_4\omega_4}{N};\ \boldsymbol{\omega}(1,2)\right)^{-1}.
\end{aligned}$$

Therefore denoting by m, n $(m<n)$ the numbers in $\{1,2,3,4\}$ different from j and l, we showed for $j,l=1,2,3,4$ with $j<l$

$$\begin{aligned}
A_{j,l} &= \prod_{1\le k_m,k_n\le N-1} S_3\left(\frac{k_m\omega_m+k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right)^{\frac12} S_3\left(\frac{k_m\omega_m+k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right)^{\frac12} \\
&\quad \times S_2\left(\frac{k_m\omega_m+k_n\omega_n}{N};\ \boldsymbol{\omega}(j,l)\right)^{-\frac12} \\
&= \prod_{1\le k_m,k_n\le N-1} S_3\left(\frac{k_m\omega_m+k_n\omega_n}{N};\ \boldsymbol{\omega}(j)\right)^{\frac12} S_3\left(\frac{k_m\omega_m+k_n\omega_n}{N};\ \boldsymbol{\omega}(l)\right)^{\frac12}.
\end{aligned}$$

This proves (4.3).

In the product $\prod_{j,l} A_{j,l}$, which is the product over terms with just two k_j's are zero, the contribution from $S_3(\cdot;\ \boldsymbol{\omega}(4))$ is calculated by Theorem 3.2 as

$$\prod_{\substack{1\le i,j\le 3\\ i\ne j}} \prod_{k_1=1}^{N-1}\prod_{k_2=1}^{N-1} S_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\ \boldsymbol{\omega}(4)\right)^{\frac32}$$

$$= \left(\frac{N^{-\frac{9}{4}+\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}} \prod_{j=1}^{3}\prod_{k=1}^{N-1} \Gamma_3\left(\frac{k\omega_j}{N}; \boldsymbol{\omega}(4)\right)^4}{\prod_{\substack{1\le i,j\le 3\\ i\neq j}} \prod_{k=1}^{N-1} \Gamma_2\left(\frac{k\omega_j}{N}; (\omega_i,\omega_j)\right) \prod_{i<j} \prod'_{0\le k_1,k_2\le N-1} \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N}; \boldsymbol{\omega}(4)\right)^2} \right)^{\frac{3}{2}}$$

$$= \frac{N^{-\frac{27}{8}+\frac{1}{8}\sum_{i\neq j}\frac{\omega_i}{\omega_j}} \prod_{j=1}^{3}\prod_{k=1}^{N-1} \Gamma_3\left(\frac{k\omega_j}{N}; \boldsymbol{\omega}(4)\right)^6}{\prod_{\substack{1\le i,j\le 3\\ i\neq j}} \prod_{k=1}^{N-1} \Gamma_2\left(\frac{k\omega_j}{N}; (\omega_i,\omega_j)\right)^{\frac{3}{2}} \prod_{1\le i<j\le 3} \prod'_{0\le k_1,k_2\le N-1} \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N}; \boldsymbol{\omega}(4)\right)^3}$$

Taking a product of such expression leads to the second identity. ∎

Theorem 4.3 *The partial product A_j is expressed in terms of the triple sine functions as*

$$A_j = \prod_{1\le k_1,k_2,k_3\le N-1} S_3\left(\frac{k_1\omega_l+k_2\omega_m+k_3\omega_n}{N};\ \boldsymbol{\omega}(j)\right)^{\frac{1}{2}}, \tag{4.4}$$

where $l<m<n$ are the elements not equal to j in $\{1,2,3,4\}$. In the product (1.2) for $r=4$, the contribution from terms with only one of the coefficients k_j being zero is

$$\prod_{j=1}^{4} A_j = N^{\frac{7}{2}-\frac{1}{12}\sum_{\substack{1\le i,j\le 4\\ i\neq j}}\frac{\omega_i}{\omega_j}} \frac{\prod'_{0\le k_1,k_2\le N-1} \prod_{l=1}^{4} \prod_{\substack{i<j\\ i,j\neq l}} \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N}; \boldsymbol{\omega}(l)\right)}{\prod_{k=0}^{N-1}\prod_{l=1}^{4}\prod_{\substack{1\le j,l\le 4\\ j\neq l}} \Gamma_3\left(\frac{k\omega_j}{N}; \boldsymbol{\omega}(l)\right)}.$$

Proof. We first calculate A_4.

$$\begin{aligned} A_4^2 &= \prod_{1\le k_1,k_2,k_3\le N-1} S_4\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right)^2 \\ &= \prod_{1\le k_1,k_2,k_3\le N-1} S_4\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right) S_4\left(\frac{(N-k_1)\omega_1+(N-k_2)\omega_2+(N-k_3)\omega_3}{N};\ \boldsymbol{\omega}\right) \\ &= \prod_{1\le k_1,k_2,k_3\le N-1} S_4\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right) S_4\left(\omega_1+\omega_2+\omega_3-\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}\right) \\ &= \prod_{1\le k_1,k_2,k_3\le N-1} S_3\left(\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{N};\ \boldsymbol{\omega}(4)\right). \end{aligned}$$

where we used the formula [KK1, Theorem 2.1]

$$S_r(z+\omega_i;\ \boldsymbol{\omega}) = S_r(z;\ \boldsymbol{\omega}) S_{r-1}(z;\ \boldsymbol{\omega}(i))^{-1}$$

and the functional equation

$$S_r(z;\ \boldsymbol{\omega}) = S_r(|\boldsymbol{\omega}| - z;\ \boldsymbol{\omega})^{(-1)^{r-1}}.$$

Hence we proved.

$$A_4 = \prod_{1 \le k_1, k_2, k_3 \le N-1} S_3\left(\frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{N};\ \boldsymbol{\omega}(4)\right)^{\frac{1}{2}}.$$

This proves (4.4) for $j = 4$. Theorem 3.3 gives

$$A_4 = N^{\frac{7}{8} - \frac{1}{24} \sum\limits_{\substack{i<j \\ 1 \le i,j \le 3}} \frac{\omega_i}{\omega_j}} \left(\frac{\prod'\limits_{0 \le k_1, k_2 \le N-1} \prod\limits_{\substack{i<j \\ 1 \le i,j \le 3}} \Gamma_3\left(\frac{k_1\omega_i + k_2\omega_j}{N}; \boldsymbol{\omega}(4)\right)}{\prod\limits_{k=1}^{N-1} \prod\limits_{l=1}^{3} \Gamma_3\left(\frac{k\omega_l}{N}; \boldsymbol{\omega}(4)\right)} \right). \tag{4.5}$$

Similarly we have A_j for $j = 1, 2, 3, 4$. Taking a product leads to the conclusion. ∎

Theorem 4.4 *In the product* (1.2) *for* $r = 4$*, the contribution from terms with none of the coefficients* k_j *being zero is given by*

$$A = 1.$$

Proof. By considering the product of (4.4), (4.3), (4.2) over all possible combinations $j < l < m$, we get all possible values $S_1(x, \omega)$, $S_2(x; \boldsymbol{\omega}')$ and $S_3(x; \boldsymbol{\omega}'')$ for any possible one, two and three dimensional vectors ω, $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ and for any possible linear combinations $x = \sum k_j\omega_j/N$ with $0 \le k_j \le N-1$. Thus by using the formula (1.2) for $r = 1, 2, 3$

$$\begin{aligned}
\prod_{j=1}^{4} A_j \prod_{\substack{j,l=1 \\ j<l}}^{4} A_{j,l} \prod_{\substack{j,l,m=1 \\ j<l<m}}^{4} A_{j,l,m} &= \left(\prod_{j=1}^{4} \prod_{\substack{0 \le k_l \le N-1 \\ l \ne j}} S_3\left(\sum_{l \ne j} \frac{k_l\omega_l}{N};\ \boldsymbol{\omega}(j)\right)\right)^{\frac{1}{2}} \\
&\quad \left(\prod_{\substack{j,l=1 \\ j<l}}^{4} \prod_{\substack{0 \le k_m \le N-1 \\ m \ne j,l}} S_2\left(\sum_{m \ne j,l} \frac{k_m\omega_m}{N};\ \boldsymbol{\omega}(j,l)\right)\right)^{-\frac{1}{2}} \\
&\quad \left(\prod_{\substack{j,l,m=1 \\ j<l<m}}^{4} \prod_{\substack{0 \le k_n \le N-1 \\ n \ne j,l,m}} S_1\left(\frac{k_n\omega_n}{N};\ \boldsymbol{\omega}(j,l,m)\right)\right)^{\frac{1}{2}} \\
&= \left(N^{\binom{4}{3}} N^{-\binom{4}{2}} N^{\binom{4}{1}}\right)^{\frac{1}{2}} \\
&= \left(N^{4-6+4}\right)^{\frac{1}{2}} \\
&= N.
\end{aligned}$$

Then by (4.1) and (1.2) we have

$$N = AN,$$

which proves $A = 1$.∎

5 Proof of Main Theorem

Proposition 5.1

$$\prod_{1\le k_1,k_2\le N-1}\prod_{1\le i<j\le 3} S_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)$$
$$= \frac{N^{-\frac{9}{4}+\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{i\neq j}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)^2}{\left(\prod_{\substack{1\le i,j\le 3\\ i\neq j}}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)\left(\prod_{i<j}\prod'_{0\le k_1,k_2\le N-1}\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)^2\right)}.$$

Proof. By Theorem 3.2,

$$\prod_{1\le k_1,k_2\le N-1}\prod_{1\le i<j\le 3} S_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)$$
$$= \prod_{i\neq j}\exp\Bigg(2\zeta_3'(0;\boldsymbol{\omega}_i^N)-\zeta_3'(0;\boldsymbol{\omega}_4^{(N)})-\zeta_3'(0,\boldsymbol{\omega})+\zeta_2'(0;(N\omega_i,\omega_j))-\zeta_2'(0;(\omega_i,\omega_j))$$
$$+(\log N)\left(2\zeta_3(0;\boldsymbol{\omega}_i^N)-\zeta_3(0;\boldsymbol{\omega}_4^{(N)})+\zeta_2(0;(N\omega_i,\omega_j))-\frac{\zeta_2(0;(\omega_i,\omega_j))}{2}\right)\Bigg).$$

In the last line the coefficient of $(\log N)$ is calculated by Theorem 2.1 as

$$\sum_{i\neq j}\Bigg(-\frac{2}{24}\left(\frac{\omega_i}{N\omega_j}+\frac{\omega_j}{\omega_4}+\frac{N\omega_4}{\omega_i}+\frac{N\omega_j}{N\omega_i}+\frac{\omega_4}{\omega_j}+\frac{\omega_i}{N\omega_4}+3\right)$$
$$+\frac{1}{24}\left(\frac{\omega_i}{\omega_j}+\frac{\omega_j}{N\omega_4}+\frac{N\omega_4}{\omega_i}+\frac{\omega_j}{N\omega_i}+\frac{N\omega_4}{\omega_j}+\frac{\omega_i}{N\omega_4}+3\right)$$
$$+\frac{1}{12}\left(N\frac{\omega_i}{\omega_j}+\frac{\omega_j}{N\omega_i}+3\right)-\frac{1}{24}\left(\frac{\omega_i}{\omega_j}+\frac{\omega_j}{\omega_i}+3\right)\Bigg)$$
$$= -\frac{4}{24}\sum_{j=1}^{3}\left(\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}\right).$$

Other parts are treated by Theorem 2.2 and Lemmas in Section 2 as

$$
\begin{aligned}
&\prod_{i\neq j}\frac{\rho_3(\omega_i,\omega_j,N\omega_4)}{\rho_3(\omega_i,\omega_j,\omega_4)}\left(\frac{\rho_3(\omega_i,\omega_j,\omega_4)}{\rho_3(\omega_i,N\omega_j,N\omega_4)}\right)^2\frac{\rho_2(\omega_i,N\omega_j)}{\rho_2(\omega_i,\omega_j)}\left(\frac{\rho_2(N\omega_j,N\omega_4)}{\rho_2(\omega_j,\omega_4)}\right)^2 N\\
&= N^{-\frac{21}{4}+\frac{2}{24}\left(\sum_{i\neq j}\left(\frac{\omega_i}{\omega_j}+\frac{\omega_j}{N\omega_4}+\frac{N\omega_4}{\omega_j}\right)\right)}\prod_{i\neq j}\prod_{0\leq k_1,k_2\leq N-1}{}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)^{-1}\\
&\quad\times\left(N^{\frac{21}{4}-\frac{1}{24}\left(\sum_{i\neq j}\left(\frac{\omega_j}{\omega_4}+\frac{N\omega_j}{\omega_i}+\frac{N\omega_4}{\omega_i}+\frac{\omega_i}{N\omega_j}+\frac{\omega_i}{\omega_4}\right)\right)}\right)^2\prod_{k=1}^{N-1}\prod_{i\neq j}\Gamma_3\left(\frac{k\omega_i}{N},(\omega_i,\omega_j,\omega_4)\right)^2\\
&\quad\times N^{-\frac{9}{2}+\frac{1}{12}\left(N+\frac{1}{N}\right)\sum_{i\neq j}\frac{\omega_i}{\omega_j}}\prod_{k=1}^{N-1}\prod_{i\neq j}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)^{-1}\\
&\quad\times\left(N^{-6+\frac{2}{12}\sum_{j=1}^{3}\left(\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}+3\right)}\right)^2\times N^6\\
&= N^{-\frac{9}{4}+\frac{1}{24}\left(2\sum_{i\neq j}\frac{\omega_i}{\omega_j}+4\left(\sum_{j=1}^{3}\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}\right)\right)}\\
&\quad\times\frac{\prod_{i\neq j}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)^2}{\left(\prod_{\substack{1\leq i,j\leq 3\\ i\neq j}}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)\left(\prod_{i<j}\prod_{0\leq k_1,k_2\leq N-1}{}'\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};\boldsymbol{\omega}\right)^2\right)}
\end{aligned}
$$

as desired. ∎

Proposition 5.2

$$
\begin{aligned}
\prod_{i\neq j}S_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right) &= N^{3+\frac{1}{12}\sum_{i\neq j}\left(\frac{N\omega_4}{\omega_j}-\frac{N\omega_i}{\omega_j}+\frac{\omega_j}{N\omega_4}-\frac{\omega_j}{N\omega_i}\right)}\\
&\quad\times\prod_{k=1}^{N-1}\prod_{i\neq j}\frac{\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\Gamma_2\left(\frac{k\omega_j}{N};(\omega_j,\omega_4)\right)}{\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)^2}.
\end{aligned}
$$

Proof. By Theorem 3.1 and Lemmas in Section 2, we have

$$
\prod_{i\neq j}S_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)
$$

$$
\begin{aligned}
&= \prod_{i\neq j} \exp\Bigg(2\Big(\zeta_3'(0;(\omega_i,\omega_j,\omega_4)) - \zeta_3'(0;(\omega_i,N\omega_j,N\omega_4))\Big) \\
&\quad +\zeta_2'(0;(\omega_i,\omega_j)) + \zeta_2'(0;(\omega_i,\omega_4)) - \zeta_2'(0;(\omega_i,N\omega_j)) - \zeta_2'(0;(\omega_i,N\omega_4)) \\
&\quad -(\log N)\Big(2\zeta_3(0;(\omega_i,N\omega_j,N\omega_4)) + 2\zeta_2(0;(\omega_i,N\omega_j)) + \zeta(0)\Big)\Bigg) \\
&= \prod_{i\neq j}\left(\frac{\rho_3(\omega_i,N\omega_j,N\omega_4)}{\rho_3(\omega_i,\omega_j,\omega_4)}\right)^2 \frac{\rho_2(\omega_i,\omega_j)}{\rho_2(\omega_i,N\omega_j)}\frac{\rho_2(\omega_i,\omega_4)}{\rho_2(\omega_i,N\omega_4)}\left(\frac{\rho_2(\omega_j,\omega_4)}{\rho_2(N\omega_j,N\omega_4)}\right)^2 N^{-1} \\
&\quad \times N^{\frac{2}{24}\left(\frac{\omega_i}{N\omega_j}+\frac{\omega_j}{\omega_4}+\frac{N\omega_4}{\omega_i}+\frac{N\omega_j}{\omega_i}+\frac{\omega_4}{\omega_j}+\frac{\omega_i}{N\omega_4}+3\right)-\frac{2}{12}\left(\frac{\omega_i}{N\omega_j}+\frac{N\omega_j}{\omega_i}+3\right)+\frac{1}{2}} \\
&= N^{-\frac{21}{2}+\frac{2}{24}\left(\sum_{i\neq j}\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}+N\left(\frac{\omega_j}{\omega_i}+\frac{\omega_4}{\omega_i}\right)+\frac{1}{N}\left(\frac{\omega_i}{\omega_j}+\frac{\omega_i}{\omega_4}\right)\right)} \prod_{i\neq j}\prod_{k=1}^{N-1}\Gamma_3\left(\frac{k\omega_i}{N};(\omega_i,\omega_j,\omega_4)\right)^{-2} \\
&\quad \times N^{9-\frac{1}{12}\sum_{i\neq j}\left(N+\frac{1}{N}\right)\left(\frac{\omega_i}{\omega_j}+\frac{\omega_4}{\omega_j}\right)} \prod_{i\neq j}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_i}{N};(\omega_i,\omega_j)\right)\Gamma_2\left(\frac{k\omega_j}{N};(\omega_j,\omega_4)\right) \\
&\quad \times \left(N^{6-\frac{2}{12}\sum_{j=1}^{3}\left(\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}+3\right)}\right)^2 N^{-6} \\
&\quad \times N^{\frac{1}{12}\sum_{i\neq j}\left(\frac{1}{N}\left(\frac{\omega_i}{\omega_4}-\frac{\omega_i}{\omega_j}\right)+N\left(\frac{\omega_4}{\omega_i}-\frac{\omega_j}{\omega_i}\right)+\frac{\omega_j}{\omega_4}+\frac{\omega_4}{\omega_j}+3\right)} \\
&= N^{3+\frac{1}{12}\sum_{i\neq j}\left(N\left(\frac{\omega_4}{\omega_j}-\frac{\omega_i}{\omega_j}\right)+\frac{1}{N}\left(\frac{\omega_j}{\omega_4}-\frac{\omega_j}{\omega_i}\right)\right)} \prod_{i\neq j}\prod_{k=1}^{N-1}\frac{\Gamma_2\left(\frac{k\omega_i}{N};(\omega_i,\omega_j)\right)\Gamma_2\left(\frac{k\omega_j}{N};(\omega_j,\omega_4)\right)}{\Gamma_3\left(\frac{k\omega_i}{N};(\omega_i,\omega_j,\omega_4)\right)^2}
\end{aligned}
$$

as desired. ∎

Proposition 5.3

$$
\begin{aligned}
\prod_{j=1}^{3} A_{j,4} &= N^{-\frac{9}{4}+\frac{1}{12}\sum_{i\neq j}\frac{\omega_i}{\omega_j}} \prod_{i\neq j}\prod_{k=1}^{N-1}\left(\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)\Gamma_3\left(\frac{k\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)\right)^2 \\
&\quad \left(\prod_{\substack{1\leq i,j\leq 3\\ i\neq j}}\prod_{k=1}^{N-1}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)\right)^{-1} \\
&\quad \left(\prod_{i<j}\ \prod_{0\leq k_1,k_2\leq N-1}{}' \Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)\Gamma_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)\right)^{-1}.
\end{aligned}
$$

Proof. By (4.3) we have

$$\prod_{j=1}^{3} A_{j,4}^2 = \prod_{1\le k_1,k_2\le N-1} \prod_{1\le i<j\le 3} S_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right) S_3\left(\frac{k_1\omega_i+k_2\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right). \tag{5.1}$$

The product of the first and the second factors in (5.1) is given by Theorem 3.2 and Proposition 5.1. Proposition follows. ∎

Proposition 5.4

$$\prod_{i<j} A_{i,j,4} = N^{\frac{3}{2}+\frac{1}{24}\left(N\sum_{i\ne j}\left(\frac{\omega_4}{\omega_j}-\frac{\omega_j}{\omega_4}\right)+\frac{1}{N}\sum_{i\ne j}\left(\frac{\omega_i}{\omega_4}-\frac{\omega_i}{\omega_j}\right)\right)}$$

$$\times \prod_{k=1}^{N-1} \frac{\prod_{i\ne j}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_j,\omega_4)\right)^{\frac{1}{2}}\Gamma_2\left(\frac{k\omega_j}{N};(\omega_i,\omega_j)\right)}{\left(\prod_{i\ne j}\Gamma_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)\right)\left(\prod_{i\ne j}\Gamma_3\left(\frac{k\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)\right)}$$

Proof. By (4.2) we deduce

$$\begin{aligned}\prod_{i<j} A_{i,j,4} &= \prod_{j=1}^{3}\prod_{k=1}^{N-1} S_4\left(\frac{k\omega_j}{N}\right)\\ &= N^{-\frac{3}{4}}\prod_{k=1}^{N-1}\left(\prod_{i\ne j} S_3\left(\frac{k\omega_j}{N};(\omega_i,\omega_j,\omega_4)\right)\prod_{j=1}^{3} S_3\left(\frac{k\omega_j}{N};(\omega_1,\omega_2,\omega_3)\right)\right)^{\frac{1}{2}},\end{aligned}$$

where the last products are calculated by Proposition 5.2 and Theorem 3.1. The proof is complete. ∎

Proof of Theorem 1.2. Each factor consisting $\widetilde{A}_4$ is calculated in (4.5) and Propositions 5.3-5.4. Taking a product leads to the conclusion. ∎

6 Special Values

In this section we present some numerical datum which are necessary for the examples given in the preceding sections.

Notation. When $\boldsymbol{\omega} = \mathbf{1} = (1, ..., 1)$, we simply write $\Gamma_r(x, \mathbf{1}) = \Gamma_r(x)$.

Lemma 6.1 *We have* $\Gamma_2(1) = e^{\zeta'(-1)}$, $\Gamma_2(2) = e^{\zeta'(-1)+\frac{1}{2}}$, $\Gamma_3(1) = e^{\frac{\zeta'(-2)+\zeta'(-1)}{2}}$, *and* $\Gamma_3(2) = e^{\frac{\zeta'(-2)-\zeta'(-1)}{2}}$. *For* $n \geq 3$, *it holds that*

$$\Gamma_2(n) = \exp\left(\zeta'(-1) + \frac{n-1}{2}\right) \prod_{k=1}^{n-2} k^{k+1-n},$$

and that

$$\Gamma_3(n) = \exp\left(\frac{\zeta'(-2) - (2n-3)\zeta'(-1)}{2} - \frac{(n-1)(n-2)}{4}\right) \prod_{k=1}^{n-1} k^{\frac{(n-k+1)(n-k+2)}{2}}.$$

Proof. For an integer $n \geq 1$, we compute

$$\zeta_r(s, n, \mathbf{1}) = \sum_{\mathbf{m} \geq 0} (\mathbf{m} \cdot \mathbf{1} + n)^{-s} = \sum_{k=n}^{\infty} \frac{f_r(k-n)}{k^s},$$

where

$$f_r(n) = \#\{(k_1, ..., k_r) \mid k_1 + \cdots + k_r = n,\ k_1, ..., k_r \geq 0\} = {}_{n+r-1}C_n.$$

When $r = 2$ or 3, since

$$f_r(n) = \begin{cases} n+1 & (r=2) \\ \frac{(n+1)(n+2)}{2} & (r=3) \end{cases},$$

we have

$$\begin{aligned} \zeta_2(s, n, \mathbf{1}) &= \sum_{k=n}^{\infty} \frac{k-n+1}{k^s} \\ &= \zeta(s-1) - (n-1)\zeta(s) - \sum_{k=1}^{n-1} \frac{k-n+1}{k^s}, \end{aligned}$$

and

$$\begin{aligned} \zeta_3(s, n, \mathbf{1}) &= \sum_{k=n}^{\infty} \frac{(k-n+1)(k-n+2)}{2k^s} \\ &= \sum_{k=n}^{\infty} \frac{k^2 - (2n-3)k + (n-1)(n-2)}{2k^s} \\ &= \frac{\zeta(s-2)}{2} - \frac{2n-3}{2}\zeta(s-1) + \frac{(n-1)(n-2)}{2}\zeta(s) - \sum_{k=1}^{n-1} \frac{(k-n+1)(k-n+2)}{2k^s}. \end{aligned}$$

Hence

$$\zeta_r'(0,n,\mathbf{1}) = \begin{cases} \zeta'(-1) + \frac{n-1}{2} + \sum_{k=1}^{n-2}(k+1-n)\log k & (r=2) \\ \frac{1}{2}\zeta'(-2) - \frac{2n-3}{2}\zeta'(-1) - \frac{(n-1)(n-2)}{4} + \sum_{k=1}^{n-1}\frac{(n+1-k)(n+2-k)}{2}\log k. & (r=3) \end{cases}$$

Thus we have the desired result as $\Gamma_r(n) = e^{\zeta_r'(0,n,\mathbf{1})}$. ■

Lemma 6.2

$$\Gamma_2\left(\frac{1}{2}\right) = 2^{-\frac{7}{24}}e^{-\frac{\zeta'(-1)}{2}}.$$

Proof. Putting $r=2$, $\boldsymbol{\omega}=\mathbf{1}$ and $N_1=N_2=2$ in Lemma 2.2, we have

$$\frac{\rho_2(1,1)}{\rho_2(\frac{1}{2},\frac{1}{2})} = \Gamma_2\left(\frac{1}{2}\right)^2\Gamma_2(1) = \Gamma_2\left(\frac{1}{2}\right)^2 e^{\zeta'(-1)}$$

by Lemma 6.1. Due to Lemma 2.1 the left hand side is equal to

$$\left(\frac{1}{2}\right)^{1-\frac{5}{12}} = 2^{-\frac{7}{12}}.$$

Hence we have the conclusion. ▮

Lemma 6.3

$$\Gamma_3\left(\frac{1}{2}\right) = 2^{-\frac{11}{48}}\exp\left(-\frac{3}{8}\zeta'(-2) - \frac{1}{2}\zeta'(-1)\right).$$

Proof. Putting $r=3$, $\boldsymbol{\omega}=\mathbf{1}$ and $N_1=N_2=N_3=2$ in Lemma 2.1, we have

$$\frac{\rho_3(1,1,1)}{\rho_3(\frac{1}{2},\frac{1}{2},\frac{1}{2})} = \Gamma_3\left(\frac{1}{2}\right)^3\Gamma_3(1)^3\Gamma_3\left(\frac{3}{2}\right).$$

From Lemma 2.1 the left hand side is equal to

$$\left(\frac{1}{2}\right)^{1-\frac{3}{8}} = 2^{-\frac{5}{8}}.$$

By Lemma 6.1 $\Gamma_3(1) = e^{\frac{\zeta'(-2)+\zeta'(-1)}{2}}$, and by Lemma 2.7 we have

$$\Gamma_3\left(\frac{3}{2}\right) = \Gamma_3\left(\frac{1}{2}\right)\Gamma_2\left(\frac{1}{2}\right)^{-1} = \Gamma_3\left(\frac{1}{2}\right)2^{\frac{7}{24}}e^{\frac{\zeta'(-1)}{2}},$$

appealing to the previous lemma. Then we have

$$\begin{aligned}
2^{-\frac{5}{8}} &= \Gamma_3\left(\frac{1}{2}\right)^4 \exp\left(3\frac{\zeta'(-2)+\zeta'(-1)}{2} + \frac{\zeta'(-1)}{2}\right) 2^{\frac{7}{24}} \\
&= \Gamma_3\left(\frac{1}{2}\right)^4 \exp\left(\frac{3\zeta'(-2)}{2} + 2\zeta'(-1)\right) 2^{\frac{7}{24}}.
\end{aligned}$$

Thus

$$\Gamma_3\left(\frac{1}{2}\right) = 2^{-\frac{11}{48}} \exp\left(-\frac{3}{8}\zeta'(-2) - \frac{1}{2}\zeta'(-1)\right).$$

∎

7 Modular Interpretations

Lemma 7.1

$$\zeta_4\left(0, \frac{\omega_1+\omega_2+\omega_3+\omega_4}{2}; \boldsymbol{\omega}\right) = \frac{7(\omega_1^4+\omega_2^4+\omega_3^4+\omega_4^4) + 10\sum_{\substack{i<j \\ 1\le i,j\le 4}} \omega_i^2\omega_j^2 - 360\omega_1\omega_2\omega_3\omega_4}{5760\omega_1\omega_2\omega_3\omega_4}.$$

Proof. We appeal to an integral representation

$$\zeta_4(s,\ x;\ \boldsymbol{\omega}) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{e^{-xt}(-t)^{s-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})(1-e^{-\omega_4 t})}dt,$$

where C is the standard coutour consisting of $+\infty \to \varepsilon > 0,\quad \varepsilon e^{i\theta}\ (0 \le \theta \le 2\pi),\quad \varepsilon \to +\infty$. By putting

$$\frac{1}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})(1-e^{-\omega_4 t})} = \frac{a_{-4}}{t^4} + \frac{a_{-3}}{t^3} + \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + O(t)$$

as $t \to 0$, we compute

$$\begin{aligned}
\zeta_4(0,\ x;\ \boldsymbol{\omega}) &= \frac{1}{2\pi i}\int_C \frac{e^{-xt}t^{-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})(1-e^{-\omega_3 t})(1-e^{-\omega_4 t})}dt \\
&= a_0 - xa_{-1} + \frac{x^2 a_{-2}}{2} - \frac{x^3 a_{-3}}{3!} + \frac{x^4 a_{-4}}{4!}.
\end{aligned}$$

We calculate each coefficient in order as follows:

$$\begin{aligned}
a_{-4} &= \frac{1}{\omega_1\omega_2\omega_3\omega_4}, \\
a_{-3} &= \frac{\omega_1+\omega_2+\omega_3+\omega_4}{2\omega_1\omega_2\omega_3\omega_4}, \\
a_{-2} &= \frac{1}{12\omega_1\omega_2\omega_3\omega_4}\left(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2+3\sum_{\substack{i<j\\1\le i,j\le 4}}\omega_i\omega_j\right), \\
a_{-1} &= \frac{1}{24\omega_1\omega_2\omega_3\omega_4}\left(\sum_{1\le i,j\le 4}\omega_i\omega_j^2+3\sum_{1\le i<j<k\le 4}\omega_i\omega_j\omega_k\right), \\
a_0 &= -\frac{1}{720\omega_1\omega_2\omega_3\omega_4}\left(\omega_1^4+\omega_2^4+\omega_3^4+\omega_4^4-5\sum_{\substack{i<j\\1\le i,j\le 4}}\omega_i^2\omega_j^2-15\sum_{i=1}^{4}\sum_{\substack{1\le j,k\le 4\\ j,k\neq i}}\omega_i^2\omega_j\omega_k\right).
\end{aligned}$$

Taking $x=\frac{\omega_1+\omega_2+\omega_3+\omega_4}{2}$ and carrying out straightforward calculations lead to the result. ∎

Proof of Theorem 1.3. By Akatsuka's product expression [A] of multiple sine functions we have

$$S_4(x;\boldsymbol{\omega}) = \exp\left(\pi i\zeta_4(0,x;\boldsymbol{\omega})+\sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e(\frac{nx}{\omega_j})}{\prod_{l\neq j}(1-e(\frac{n\omega_l}{\omega_j}))}\right). \tag{7.1}$$

When $x=(\omega_1+\omega_2+\omega_3+\omega_4)/2$, we compute

$$e\left(\frac{nx}{\omega_j}\right)=\prod_{l=1}^{4}e\left(\frac{n\omega_l}{2\omega_j}\right)=(-1)^n\prod_{l\neq j}e\left(\frac{n\omega_l}{2\omega_j}\right).$$

Hence

$$\begin{aligned}
S_4(x;\boldsymbol{\omega}) &= \exp\left(\pi i\zeta_4(0,x;\boldsymbol{\omega})+\sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{1}{n}\prod_{l\neq j}\frac{(-1)^n e\left(\frac{n\omega_l}{2\omega_j}\right)}{1-e(\frac{n\omega_l}{\omega_j})}\right) \\
&= \exp\left(\pi i\zeta_4(0,x;\boldsymbol{\omega})+\sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{1}{n}\prod_{l\neq j}\frac{(-1)^n}{e(-\frac{n\omega_l}{2\omega_j})-e(\frac{n\omega_l}{2\omega_j})}\right) \\
&= \exp\left(\pi i\zeta_4(0,x;\boldsymbol{\omega})+\sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{1}{n}\prod_{l\neq j}\frac{(-1)^n}{-2i\sin(\pi\frac{n\omega_l}{\omega_j})}\right)
\end{aligned}$$

$$= \exp\left(\pi i\zeta_4(0,x;\boldsymbol{\omega}) + \sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(-1)^n}{8i}\prod_{l\neq j}\frac{1}{\sin(\pi\frac{n\omega_l}{\omega_j})}\right).$$

When $\boldsymbol{\omega} = (\tau_1,\tau_2,\tau_3,1)$ and $x = (\tau_1+\tau_2+\tau_3+1)/2$, Theorem 4.4 implies that $S_4(x;\boldsymbol{\omega}) = 1$. Thus

$$-8\pi\zeta_4(0,x;\boldsymbol{\omega}) + \sum_{j=1}^{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\prod_{l\neq j}\frac{1}{\sin(\pi\frac{n\omega_l}{\omega_j})} = 0.$$

This leads to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sin(\pi n\tau_1)\sin(\pi n\tau_2)\sin(\pi n\tau_3)}\\
&= 8\pi\zeta_4(0,x;\boldsymbol{\omega}) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n\sin(\pi\frac{n}{\tau_1})\sin(\pi\frac{n\tau_2}{\tau_1})\sin(\pi\frac{n\tau_3}{\tau_1})}\\
&\quad -\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sin(\pi\frac{n\tau_1}{\tau_2})\sin(\pi\frac{n}{\tau_2})\sin(\pi\frac{n\tau_3}{\tau_2})} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n\sin(\pi\frac{n\tau_1}{\tau_3})\sin(\pi\frac{n\tau_2}{\tau_3})\sin(\pi\frac{n}{\tau_3})}.
\end{aligned}$$

∎

Proof of Theorem 1.4. We compute the product (7.1) for

$$\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3,\omega_4) = (\tau_1,\tau_2,\tau_3,1)$$

and

$$x = \frac{\tau_1+\tau_2+\tau_3+1}{2}.$$

The factor for $j = 4$ in the double sum is

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{e(nx)}{n\prod_{l=1}^{3}(1-e(n\tau_l))} &= \sum_{n=1}^{\infty}\frac{e(nx)}{n}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}e(n(n_1\tau_1+n_2\tau_2+n_3\tau_3))\\
&= -\log\prod_{n_1=0}^{\infty}\prod_{n_2=0}^{\infty}\prod_{n_3=0}^{\infty}(1-e(x+n_1\tau_1+n_2\tau_2+n_3\tau_3))\\
&= -\log\prod_{n_1=0}^{\infty}\prod_{n_2=0}^{\infty}\prod_{n_3=0}^{\infty}\left(1+e\left(\left(n_1+\frac{1}{2}\right)\tau_1+\left(n_2+\frac{1}{2}\right)\tau_2+\left(n_3+\frac{1}{2}\right)\tau_3\right)\right)\\
&= -\log F(\tau_1,\tau_2,\tau_3).
\end{aligned}$$

For $j = 1, 2, 3$, we can similarly deal with the terms involving ω_l/ω_j $(l < j)$ as $\mathrm{Im}(\omega_l/\omega_j) > 0$. But in the other cases when $l > j$, we have $\mathrm{Im}(\omega_l/\omega_j) < 0$ and compute as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \frac{e(\frac{nx}{\omega_j})}{\prod\limits_{l\neq j}(1 - e(\frac{n\omega_l}{\omega_j}))} &= \sum_{n=1}^{\infty} \frac{e(\frac{nx}{\omega_j})}{n} \frac{1}{\prod\limits_{l>j}(1 - e(\frac{n\omega_l}{\omega_j}))\prod\limits_{l<j}(1 - e(\frac{n\omega_l}{\omega_j}))} \\
&= \sum_{n=1}^{\infty} \frac{e(\frac{nx}{\omega_j})}{n} \prod_{l>j} \frac{e(-\frac{n\omega_l}{\omega_j})}{e(-\frac{n\omega_l}{\omega_j}) - 1} \prod_{l<j} \frac{1}{1 - e(\frac{n\omega_l}{\omega_j})} \\
&= \sum_{n=1}^{\infty} \frac{e(\frac{nx}{\omega_j})}{n} \prod_{l>j} \left(-\sum_{n_l=0}^{\infty} e\left(-\frac{n(n_l+1)\omega_l}{\omega_j} \right) \right) \prod_{l<j} \sum_{n_l=0}^{\infty} e\left(\frac{nn_l\omega_l}{\omega_j} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \prod_{l>j} \left(-\sum_{n_l=0}^{\infty} e\left(-\frac{n(n_l+\frac{1}{2})\omega_l}{\omega_j} \right) \right) \prod_{l<j} \sum_{n_l=0}^{\infty} e\left(\frac{n(n_l+\frac{1}{2})\omega_l}{\omega_j} \right) \\
&= (-1)^{\#\{l \mid l>j\}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{\substack{n_l=0 \\ l\neq j}}^{\infty} e\left(-\sum_{l>j} \frac{n(n_l+\frac{1}{2})\omega_l}{\omega_j} + \sum_{l<j} \frac{n(n_l+\frac{1}{2})\omega_l}{\omega_j} \right) \\
&= (-1)^{\#\{l \mid l>j\}+1} \sum_{\substack{n_l=0 \\ l\neq j}}^{\infty} \log\left(1 + e\left(-\sum_{l>j} \frac{(n_l+\frac{1}{2})\omega_l}{\omega_j} + \sum_{l<j} \frac{(n_l+\frac{1}{2})\omega_l}{\omega_j} \right) \right) \\
&= (-1)^{\#\{l \mid l>j\}+1} \log \prod_{\substack{n_l=0 \\ l\neq j}}^{\infty} \left(1 + e\left(-\sum_{l>j} \frac{(n_l+\frac{1}{2})\omega_l}{\omega_j} + \sum_{l<j} \frac{(n_l+\frac{1}{2})\omega_l}{\omega_j} \right) \right) \\
&= \begin{cases} \log F\left(\frac{\tau_1}{\tau_3}, \frac{\tau_2}{\tau_3}, -\frac{1}{\tau_3} \right) & (j=3) \\ -\log F\left(\frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2}, -\frac{\tau_3}{\tau_2} \right) & (j=2) \\ \log F\left(-\frac{1}{\tau_1}, -\frac{\tau_2}{\tau_1}, -\frac{\tau_3}{\tau_1} \right) & (j=1). \end{cases}
\end{aligned}$$

Hence from Theorem 4.4 and (7.1) we have

$$1 = S_4(x;\boldsymbol{\omega}) = e^{\pi i \zeta_4(0,x;\boldsymbol{\omega})} \frac{F\left(\frac{\tau_1}{\tau_3}, \frac{\tau_2}{\tau_3}, -\frac{1}{\tau_3} \right) F\left(-\frac{1}{\tau_1}, -\frac{\tau_2}{\tau_1}, -\frac{\tau_3}{\tau_1} \right)}{F(\tau_1,\tau_2,\tau_3) F\left(\frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2}, -\frac{\tau_3}{\tau_2} \right)}.$$

∎

8 The graph of $S_4(x; \boldsymbol{\omega})$

Throughout this section we assume $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbf{R}^4$ with $0 < \omega_1 \le \omega_2 \le \omega_3 \le \omega_4$, and put $S(x) = S_4(x;\boldsymbol{\omega})$ for simplicity. We recall the notation $|\boldsymbol{\omega}| := \omega_1 + \omega_2 + \omega_3 + \omega_4$.

Lemma 8.1 *At least one of the following ten special values are greater than one:*

$$S\left(\frac{\omega_j}{2}\right) \qquad (j=1,2,3,4), \qquad S\left(\frac{\omega_i+\omega_j}{2}\right) \qquad (i,j=1,2,3,4, \quad i<j).$$

Proof. By Proposition 2.4 in [KK1] we have for any $l = 1,2,3,4$

$$S_3(x;\boldsymbol{\omega}(l)) = S_4(x;\boldsymbol{\omega})S_4(x+\omega_l;\boldsymbol{\omega})^{-1} = S_4(x;\boldsymbol{\omega})S_4(|\boldsymbol{\omega}|-x-\omega_l;\boldsymbol{\omega}).$$

Taking $x = |\boldsymbol{\omega}(l)|/2$ leads to

$$S_3\left(\frac{|\boldsymbol{\omega}(l)|}{2};\boldsymbol{\omega}(l)\right) = S_4\left(\frac{|\boldsymbol{\omega}(l)|}{2};\boldsymbol{\omega}\right)^2 = S\left(\frac{|\boldsymbol{\omega}(l)|}{2}\right)^2.$$

By Kurokawa's recent result $S_3\left(\frac{|\boldsymbol{\omega}(l)|}{2};\boldsymbol{\omega}(l)\right) < 1$ proved in [K3], we have

$$S\left(\frac{|\boldsymbol{\omega}(l)|}{2}\right) < 1$$

for any $l = 1,2,3,4$. Thus in the formula (1.2) with $N = 2$

$$\left(\prod_{j=1}^{4} S\left(\frac{\omega_j}{2}\right)\right)\left(\prod_{\substack{1\le i,j\le 4\\ i<j}} S\left(\frac{\omega_i+\omega_j}{2}\right)\right)\left(\prod_{\substack{1\le i,j,k\le 4\\ i<j<k}} S\left(\frac{\omega_i+\omega_j+\omega_k}{2}\right)\right) = 2,$$

the third product in the left hand side is less than 1. Thus the remaining part must be greater than 1. Hence we have the desired conclusion. ∎

Lemma 8.2

$$\left(\frac{S'}{S}\right)''\left(\frac{|\boldsymbol{\omega}|}{2}\right) < 0.$$

Proof. By definition we have

$$S(x) = \Gamma_4(x;\boldsymbol{\omega})^{-1}\Gamma_4(|\boldsymbol{\omega}|-x;\boldsymbol{\omega})$$

with

$$\Gamma_4(x;\boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_4(0,x;\boldsymbol{\omega})\right)$$

for

$$\zeta_4(s,x;\boldsymbol{\omega}) = \sum_{n_1,n_2,n_3,n_4\ge 0}(n_1\omega_1+n_2\omega_2+n_3\omega_3+n_4\omega_4+x)^{-s}.$$

We find that $\zeta_4(s,x;\boldsymbol{\omega})$ is absolutely convergent in $\mathrm{Re}(s) > 4$ and that it has a meromorphic continuation to all $s \in \mathbf{C}$. Moreover $\zeta_4(s,x;\boldsymbol{\omega})$ is holomorphic at $s = 0$.

Since

$$\begin{aligned}\log S(x) &= -\log \Gamma_4(x;\boldsymbol{\omega}) + \log \Gamma_4(|\boldsymbol{\omega}|-x;\boldsymbol{\omega}) \\ &= -\frac{\partial}{\partial s}\zeta_4(0,x;\boldsymbol{\omega}) + \frac{\partial}{\partial s}\zeta_4(0,|\boldsymbol{\omega}|-x;\boldsymbol{\omega})\end{aligned}$$

we have

$$\frac{S'}{S}(x) = -\frac{\partial^2}{\partial s \partial x}\zeta_4(0,x;\boldsymbol{\omega}) - \frac{\partial^2}{\partial s \partial x}\zeta_4(0,|\boldsymbol{\omega}|-x;\boldsymbol{\omega}).$$

Hence

$$\left(\frac{S'}{S}\right)'(x) = -\frac{\partial^3}{\partial s \partial x^2}\zeta_4(0,x;\boldsymbol{\omega}) + \frac{\partial^3}{\partial s \partial x^2}\zeta_4(0,|\boldsymbol{\omega}|-x;\boldsymbol{\omega})$$

and differentiating repeatedly gives

$$\begin{aligned}\left(\frac{S'}{S}\right)^{(4)}(x) &= -\frac{\partial^6}{\partial s \partial x^5}\zeta_4(0,x;\boldsymbol{\omega}) - \frac{\partial^4}{\partial s \partial x^3}\zeta_4(0,|\boldsymbol{\omega}|-x;\boldsymbol{\omega}) \\ &= 4!\left(\sum_{n_1,n_2,n_3,n_4\geq 0}(n_1\omega_1+n_2\omega_2+n_3\omega_3+n_4\omega_4+x)^{-5}\right. \\ &\qquad \left. + \sum_{m_1,m_2,m_3,m_4\geq 1}(m_1\omega_1+m_2\omega_2+m_3\omega_3+m_4\omega_4-x)^{-5}\right),\end{aligned}$$

where we use the relation

$$\frac{\partial^5}{\partial x^5}\zeta_4(s,x;\boldsymbol{\omega}) = (-s)(-s-1)(-s-2)(-s-3)(-s-4)\zeta_4(s+5,x;\boldsymbol{\omega})$$

coming from

$$\begin{aligned}\frac{\partial}{\partial x}\zeta_4(s,x;\boldsymbol{\omega}) &= \frac{\partial}{\partial x}\left(\sum_{n_1,n_2,n_3,n_4\geq 0}(n_1\omega_1+n_2\omega_2+n_3\omega_3+n_4\omega_4+x)^{-s}\right) \\ &= -s\sum_{n_1,n_2,n_3,n_4\geq 0}(n_1\omega_1+n_2\omega_2+n_3\omega_3+n_4\omega_4+x)^{-s-1} \\ &= -s\zeta_4(s+1,x;\boldsymbol{\omega}).\end{aligned}$$

Thus

$$\left(\frac{S'}{S}\right)^{(4)}(x) > 0 \tag{8.1}$$

in $0 < x < |\boldsymbol{\omega}|$ and

$$\left(\frac{S'}{S}\right)^{(3)}\left(\frac{|\boldsymbol{\omega}|}{2}\right) = 0. \tag{8.2}$$

x	0	$\cdots$	$\frac{\lvert\boldsymbol{\omega}\rvert}{2}$	$\cdots$	$\lvert\boldsymbol{\omega}\rvert$
$\left(\frac{S'}{S}\right)^{(4)}$		$+$		$+$	
$\left(\frac{S'}{S}\right)^{(3)}$		$\nearrow$	0	$\nearrow$	
$\left(\frac{S'}{S}\right)^{(3)}$		$-$	0	$+$	
$\left(\frac{S'}{S}\right)''$		$\searrow$		$\nearrow$	

By (8.1) and (8.2) we see as in the above table that $\left(\frac{S'}{S}\right)^{(3)}(x) < 0$ for $0 < x < \frac{|\boldsymbol{\omega}|}{2}$ and $\left(\frac{S'}{S}\right)^{(3)}(x) > 0$ for $\frac{|\boldsymbol{\omega}|}{2} < x < |\boldsymbol{\omega}|$, and consequently that $\left(\frac{S'}{S}\right)''(x)$ takes the minimum at $x = \frac{|\boldsymbol{\omega}|}{2}$.

For proving $\left(\frac{S'}{S}\right)''\left(\frac{|\boldsymbol{\omega}|}{2}\right) < 0$, we assume $\left(\frac{S'}{S}\right)''\left(\frac{|\boldsymbol{\omega}|}{2}\right) \geq 0$ to have a contradiction as below. By the above table, the assumption $\left(\frac{S'}{S}\right)''\left(\frac{|\boldsymbol{\omega}|}{2}\right) \geq 0$ implies that $\left(\frac{S'}{S}\right)''(x) \geq 0$ for $0 < x < |\boldsymbol{\omega}|$. Hence $\left(\frac{S'}{S}\right)'(x)$ is increasing throughout the interval. Proposition 2.4 in [KK1] shows that

$$S(x) = \frac{x}{x - |\boldsymbol{\omega}|} \times O(1) \tag{8.3}$$

with $O(1)$ part being a nonzero holomorphic bounded function as $x \to 0$ and $x \to |\boldsymbol{\omega}|$. Thus

$$\lim_{x \to +0} \left(\frac{S'}{S}\right)'(x) = -\infty$$

and

$$\lim_{x \to |\boldsymbol{\omega}|} \left(\frac{S'}{S}\right)'(x) = \infty.$$

Therefore there exists a unique $\gamma \in (0, |\boldsymbol{\omega}|)$ such that $\left(\frac{S'}{S}\right)'(\gamma) = 0$. This means that $\left(\frac{S'}{S}\right)(x)$ takes the minimum at $x = \gamma$. If this minimum is non-negative, then $\left(\frac{S'}{S}\right)(x) \geq 0$ and so $S'(x) \geq 0$. This shows $S(x)$ is increasing, which contradicts the previous lemma, because Theorem 1.1 says

$$S\left(\frac{|\boldsymbol{\omega}|}{2}\right) = 1.$$

Hence the minimum is negative. Here we again use (8.3) to get

$$\lim_{x \to 0, |\boldsymbol{\omega}|} \frac{S'}{S}(x) = \infty.$$

Then there exist $\gamma_1 \in (0, \gamma)$ and $\gamma_2 \in (\gamma, |\boldsymbol{\omega}|)$ such that $\frac{S'}{S}(\gamma_1) = \frac{S'}{S}(\gamma_2) = 0$, and we see that $\frac{S'}{S}(x)$ is positive in $[0, \gamma_1) \cup (\gamma_2, |\boldsymbol{\omega}|]$, and negative in (γ_1, γ_2). As $S(x) > 0$ in the whole interval, we also determine the signuature of $S'(x)$ to obtain the behavior of $S(x)$.

Since $S(x)S(|\boldsymbol{\omega}|-x)=1$, it holds that $\log S(x)$ is symmtric with respect to the point $(|\boldsymbol{\omega}|/2,\,0)$. Thus $\gamma_1<|\boldsymbol{\omega}|/2<\gamma_2$, and consequently there exists some $c\in(0,|\boldsymbol{\omega}|/2)$ such that $0<S(x)<1$ in $x\in(0,\,c)$ and $1<S(x)$ in $x\in(c,\,|\boldsymbol{\omega}|/2)$. This implies that if two points $x,\,x'\in(0,\,|\boldsymbol{\omega}|/2)$ satisfy that $S(x)<1$ and that $S(x')>1$, then it must hold that $x<x'$. This contradicts the previous lemma, taking into account that

$$S\left(\frac{\omega_1+\omega_2+\omega_3}{2}\right)=S_4\left(\frac{\omega_1+\omega_2+\omega_3}{2};\boldsymbol{\omega}\right)=S_3\left(\frac{\omega_1+\omega_2+\omega_3}{2};(\omega_1,\omega_2,\omega_3)\right)^{\frac{1}{2}}<1,$$

which is shown by Kurokawa [K3]. Hence the conclusion. ∎

Theorem 8.1 *The function $S(x)=S_4(x;\boldsymbol{\omega})$ has four extremal values in the interval $(0,|\boldsymbol{\omega}|)$. More precisely each of the intervals $(0,|\boldsymbol{\omega}|/2)$ and $(|\boldsymbol{\omega}|/2,|\boldsymbol{\omega}|)$ has both a maximal and a minimal points.*

Proof. By the previous lemma and by (8.3) we have the following table.

x	0	$\cdots$	$\frac{\lvert\boldsymbol{\omega}\rvert}{2}$	$\cdots$	$\lvert\boldsymbol{\omega}\rvert$
$\left(\frac{S'}{S}\right)''$	$+\infty$	$\searrow$	$-$	$\nearrow$	$+\infty$
$\left(\frac{S'}{S}\right)''$		$+\ 0\ -$	$-$	$-\ 0\ +$	
$\left(\frac{S'}{S}\right)'$	$-\infty$	$\nearrow\searrow$	0	$\searrow\nearrow$	$+\infty$
$\left(\frac{S'}{S}\right)'$	$-\infty$	$-\ 0\ +$	0	$-\ 0\ +$	$+\infty$
$\frac{S'}{S}$	$+\infty$	$\searrow\nearrow$		$\searrow\nearrow$	$+\infty$

As shown in the proof of the previous lemma, $S(x)$ cannot be monotone increasing in $(0,|\boldsymbol{\omega}|/2)$. So the minimal value of $\frac{S'}{S}$ in $(0,|\boldsymbol{\omega}|/2)$ should be negative. Therefore we have the following table:

x	0	$\cdots$	$\frac{\lvert\boldsymbol{\omega}\rvert}{2}$	$\cdots$	$\lvert\boldsymbol{\omega}\rvert$
$\frac{S'}{S}$	$+\infty$	$+\ 0\ -\ 0\ +$	$+$	$+\ 0\ -\ 0\ +$	$+\infty$
S'		$+\ 0\ -\ 0\ +$	$+$	$+\ 0\ -\ 0\ +$	
S	0	$\nearrow\searrow\nearrow$	1	$\nearrow\searrow\nearrow$	$+\infty$

The proof is complete. ∎

Theorem 8.2 *Let $\alpha,\,\beta\in(0,|\boldsymbol{\omega}|/2)$ satisfy $S(\alpha)=S(\beta)=1$ with $\alpha<\beta$ as shown in Figure 1 in Section 1. Then for $0<\omega_1\leq\omega_2\leq\omega_3\leq\omega_4$, the following statements concerning the location of 2-division points are true.*

(1) At least one of $\frac{\omega_j}{2}$ $(j=1,2,3,4)$ and $\frac{\omega_i+\omega_j}{2}$ $(i,j=1,2,3,4,\ i<j)$ lie in the interval (α,β).

(2) $\frac{\omega_3+\omega_4}{2} > \alpha$.

(3) Both $\frac{\omega_1+\omega_3+\omega_4}{2}$ *and* $\frac{\omega_2+\omega_3+\omega_4}{2}$ *lie in the interval* $(\beta, |\boldsymbol{\omega}|/2)$.

Proof. The first assertion follows from Lemma 8.1 and the previous theorem.

The second assertion follows from the first one, because $\frac{\omega_3+\omega_4}{2}$ is the largest among the ten points given in (1), and in particular it is greater than the element in (α, β).

Then by (2), since both $\frac{\omega_1+\omega_3+\omega_4}{2}$ and $\frac{\omega_2+\omega_3+\omega_4}{2}$ are greater than $\frac{\omega_3+\omega_4}{2}$, they are both greater than α. On the other hand by Kurokawa's result [K3], it holds that $S\left(\frac{\omega_1+\omega_3+\omega_4}{2}\right) < 1$ and $S\left(\frac{\omega_2+\omega_3+\omega_4}{2}\right) < 1$, as shown at the end of the proof of Lemma 8.1. Hence we obtain the last assertion. ∎

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