

Absolute zeta functions, absolute Riemann hypothesis and absolute Casimir energies

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Abstract. We prove that the absolute zeta functions for finite abelian monoids have positive Casimir energy if and only if its order is divisible by 4.

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1 Introduction

In 1948, Casimir used the value

$$\zeta(-1) = -\frac{1}{12} \tag{1.1}$$

for the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for calculating a certain force between two metal objects; see Casimir [C], Hawking [Haw], and Kurokawa-Wakayama [KW1]; we remark that $\zeta(-1)$ appears in the one dimensional case and that $\zeta(-3)$ is used in the ordinary three dimensional situation. The value (1.1) represents the value of a divergent series

$$1 + 2 + 3 + \cdots .$$

When a zeta function

$$Z(s) = \sum_{\lambda} \lambda^{-s}$$

is given, we call $Z(-1)$ its Casimir energy, which is named after the work of Casimir.

From various viewpoints, the signature of the Casimir energy is meaningful in the sense that it reflects certain dynamical and arithmetical properties. For example, the following proposition is known.

Proposition 1.1. *Let K be a finite extension of $\mathbf{Q}$ with its integer ring O_K. The Casimir energy for the Dedekind zeta function*

$$\zeta_K(s) = \sum_{\substack{I \subset O_K \\ \text{ideal} \neq 0}} N(I)^{-s}$$

with $N(I) = |O_K/I|$ is as follows:

$$\zeta_K(-1) \text{ is } \begin{cases} \text{positive} & \Longleftrightarrow K \text{ is totally real and } [K:\mathbf{Q}] \text{ is even,} \\ \text{negative} & \Longleftrightarrow K \text{ is totally real and } [K:\mathbf{Q}] \text{ is odd,} \\ 0 & \Longleftrightarrow K \text{ is not totally real.} \end{cases}$$

For example, $\zeta(-1) = -1/12$ is actually negative for $K = \mathbf{Q}$.

In this paper, we prove an analog of Proposition 1 for the absolute Weil zeta function. We recall briefly the history of absolute mathematics. The absolute mathematics has been studied in many papers after Tits[T] (1957) as in the papers listed in the references. It is recommended to read the excellent survey of Manin [M] published in 1995. Early publications are [K1](1992) and [KOW](2003); see also [CC], [D], [Har], [K2], [KO], [KW2], and [S]. The absolute zeta functions were studied recently in papers [DKK] and [KKK]. Here we describe the definitions:

Let $\boldsymbol{\mu}_N = \{z \in \mathbf{C} \mid z^N = 1\}$ $(N = 1, 2, 3, ...)$ and put

$$\mathbf{F}_{1^N} = \boldsymbol{\mu}_N \cup \{0\}.$$

The absolute Weil zeta function is defined in [DKK] by

$$\zeta_{\mathbf{F}_{1^N}}(s) = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(\mathbf{F}_{1^N}, \mathbf{F}_{1^m})|}{m} e^{-ms}\right).$$

More generally, for a fintie abelian group G, we put $\mathbf{F}_1[G] = G \cup \{0\}$, which is a multiplicative monoid (an $\mathbf{F}_1$-algebra). We define the absolute Weil zeta function of $\mathbf{F}_1[G]$ by

$$\zeta_{\mathbf{F}_1[G]} = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(\mathbf{F}_1[G], \mathbf{F}_{1^m})|}{m} e^{-ms}\right).$$

In particular, we have $\mathbf{F}_{1^N} = \mathbf{F}_1[\boldsymbol{\mu}_N]$.

These absolute zeta functions are constructed for $\mathbf{F}_1$-algebras, where $\mathbf{F}_1$ is “the field with one element". Precisely speaking we understand that “an $\mathbf{F}_1$-algebra" means “a multiplicative monoid with 0". Especially $\mathbf{F}_1 = \{1, 0\}$.

These absolute zeta functions are considered as absolute version of the classical Weil (or congruence) zeta functions. In the classical case, we are looking at the number of rational points over finite fields. In the absolute case, we look at the number of rational points over finite extensions of $\mathbf{F}_1$.

In general they satisfy analytic continuations, functional equations and the associated Riemann hypothesis. Special values at negative integers are considered to be absolute Casimir energies. Here we restrict to the zeta function of a finite extension of $\mathbf{F}_1$.

For describing our earlier results, we introduce the absolute Frobenius operator Φ_G for a finite abelian group G, which operates on $\boldsymbol{\mu}_{N^2}$ with $|G| = N$. First when $G = \boldsymbol{\mu}_N$, we define

$$\Phi_G : \boldsymbol{\mu}_{N^2} \ni \alpha \longmapsto \alpha^{N+1} \in \boldsymbol{\mu}_{N^2}.$$

For general cases, put

$$G \cong \boldsymbol{\mu}_{N_1} \times \cdots \times \boldsymbol{\mu}_{N_r}$$

with $N = N_1 \cdots N_r$. Then

$$\Phi_G = \Phi_{N_1,\ldots,N_r} : \boldsymbol{\mu}_{N_1^2} \times \cdots \times \boldsymbol{\mu}_{N_r^2} \longrightarrow \boldsymbol{\mu}_{N_1^2} \times \cdots \times \boldsymbol{\mu}_{N_r^2}$$

is defined by the Kronecker tensor product

$$\Phi_G = \Phi_{N_1} \otimes \cdots \otimes \Phi_{N_r}.$$

In [DKK] and [KKK], we proved the following theorem:

Theorem 1.2. *For any finite abelian group G, the following properties hold:*

(i) $\zeta_{\mathbf{F}_1[G]}(s)$ *has analytic continuation to all the complex numbers.*

(ii) $\zeta_{\mathbf{F}_1[G]}(s)$ *satisfies the associated functional equation*

$$\zeta_{\mathbf{F}_1[G]}(-s) = w(G)e^{-|G|s}\zeta_{\mathbf{F}_1[G]}(s),$$

where $w(G)$ is a complex number of modulus 1.

(iii) $\zeta_{\mathbf{F}_1[G]}(s)$ *satisfies the analogue of the Riemann hypothesis. Namely, all singularities of $\zeta_{\mathbf{F}_1[G]}(s)$ are on the line* $\mathrm{Re}(s) = 0$.

(iv) $\zeta_{\mathbf{F}_1[G]}(s)$ *has the determinant expression*

$$\zeta_{\mathbf{F}_1[G]}(s) = \det\left(1 - \Phi_G e^{-s}\right)^{-1/|G|}. \tag{1.2}$$

We define the normalized Casimir energy for the absolute zeta function of an $\mathbf{F}_1$-algebra A by

$$\zeta_A(-1)^{|\mathrm{End}(A)|}.$$

Our main result is as follows:

Theorem 1.3. *Let A be a fintie abelian $\mathbf{F}_1$-algebra. Then we have*

$$\zeta_A(-1)^{|\mathrm{End}(A)|} > 0 \Longleftrightarrow 4 \mid [A : \mathbf{F}_1].$$

2 Proofs

We start by proving a special case where $A = \mathbf{F}_{1^N}$. The following Lemma will be useful in the proof.

Lemma 2.1. *The integer*

$$a(N) = \sum_{n|N} \frac{N}{n}\varphi(n)$$

is even, if and only if $4|N$.

Proof. The sequence $a(N)$ is a convolution of $b(N) = N$ and $c(N) = \varphi(N)$, since

$$(b * c)(N) = \sum_{mn=N} b(m)c(n) = \sum_{n|N} b\left(\frac{N}{n}\right) c(n) = \sum_{n|N} \frac{N}{n}\varphi(n) = a(N).$$

Because both $b(N)$ and $c(N)$ are multiplicative, the convolution $a(N)$ is also multiplicative. Therefore

$$a(N) = \prod_{\substack{p|N \\ p\text{: prime}}} a(p^{\mathrm{ord}_p(N)}).$$

Hence $a(N)$ is odd if and only if $a(p^{\mathrm{ord}_p(N)})$ is odd for all $p|N$. By definition it holds that $a(1) = 1$ and

$$a(p^l) = \sum_{k=0}^{l} p^{l-k}\varphi(p^k). \tag{2.1}$$

We will examine the condition for (2.1) being odd. When p is odd, we compute

$$\begin{aligned} a(p^l) &= p^l\varphi(1) + \sum_{k=1}^{l} p^{l-k}\varphi(p^k) \\ &= p^l + \sum_{k=1}^{l} p^{l-k}p^{k-1}(p-1) \\ &= p^l + (p-1)p^{l-1}l, \end{aligned}$$

which is always odd. When $p = 2$, we compute

$$\begin{aligned} a(2^l) &= 2^l + 2^{l-1}l \\ &= \begin{cases} \text{odd} & (l = 1) \\ \text{even} & (l \geq 2). \end{cases} \end{aligned}$$

Consequently, (2.1) is odd if and only if either p is odd or $(l, p) = (1, 2)$. In other words, (2.1) is even if and only if $\mathrm{ord}_2(N) \geq 2$, which equivalently is $4|N$. □

Proposition 2.2.
$$\zeta_{\mathbf{F}_{1^N}}(-1)^N > 0 \Longleftrightarrow 4 \mid N.$$

Proof. By [DKK] we have an Euler product expression

$$\zeta_{\mathbf{F}_{1^N}}(s)^N = \prod_{n|N}(1-e^{-ns})^{-\frac{N}{n}\varphi(n)}.$$

By this expression we have

$$\begin{aligned}\operatorname{sgn}(\zeta_{\mathbf{F}_{1^N}}(-1)^N) &= (-1)^{\sum_{n|N}\frac{N}{n}\varphi(n)} \\ &= \begin{cases} 1 & \text{if } \sum_{n|N}\frac{N}{n}\varphi(n) \text{ is even,} \\ -1 & \text{if } \sum_{n|N}\frac{N}{n}\varphi(n) \text{ is odd.}\end{cases}\end{aligned}$$

Then by the previous lemma, it holds that

$$\operatorname{sgn}(\zeta_{\mathbf{F}_{1^N}}(-1)^N) = \begin{cases} 1 & \text{if } 4|N, \\ -1 & \text{if } 4 \not| N.\end{cases}$$

□

Proof of Theorem 1:

By the determinant expression (1.2) proved in [KKK], we have

$$\begin{aligned}\zeta_{\mathbf{F}_{1^N}}(-1)^N &= \det(1-\mathbf{\Phi}_N e)^{-1} \\ &= \det((-\mathbf{\Phi}_N e)(\mathbf{\Phi}_N^{-1}e^{-1}-1))^{-1} \\ &= (-1)^{N^2}\det(\mathbf{\Phi}_N)e^{-N^2}\det(1-\mathbf{\Phi}_N e^{-1})^{-1} \\ &= (-1)^{N^2}\det(\mathbf{\Phi}_N)e^{-N^2}\zeta_{\mathbf{F}_{1^N}}(1)^N \\ &= (-1)^{N^2}\det(\mathbf{\Phi}_N)e^{-N^2}\prod_{n|N}(1-e^{-n})^{-\frac{N}{n}\varphi(n)}.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{sgn}(\zeta_{\mathbf{F}_{1^N}}(-1)^N) &= (-1)^{N^2}\det(\mathbf{\Phi}_N) \\ &= (-1)^{N^2}\operatorname{sgn}(\mathbf{\Phi}_N).\end{aligned}$$

By the preceding proposition, this is negative if and only if $4 \not| N$. Thus

$$\operatorname{sgn}(\mathbf{\Phi}_N) = \begin{cases} 1 & \text{if } N \equiv 1,3 \pmod 4, \\ -1 & \text{if } N \equiv 2 \pmod 4.\end{cases} \tag{2.2}$$

By [KKK] we have the determinant expression

$$\zeta_{\mathbf{F}_1[G]}(s)^N = \det(1-\Phi_G e^{-s})^{-1}.$$

Thus we compute

$$\begin{aligned}\operatorname{sgn}(\zeta_{\mathbf{F}_1[G]}(-1)^N) &= (-1)^{N^2}\det(\Phi_{N_1,\ldots,N_r}) \\ &= (-1)^{N^2}\operatorname{sgn}(\Phi_{N_1})^{N_2^2\cdots N_r^2}\cdots\operatorname{sgn}(\Phi_{N_r})^{N_1^2\cdots N_{r-1}^2}.\end{aligned}$$

First when all of $N_1, ..., N_r$ are odd, we find by (2.2) that $\operatorname{sgn}(\zeta_{\mathbf{F}_1[G]}(-1)^N) = -1$.

Next when $2|N_1\cdots N_r$ but $4 \not| N_1\cdots N_r$, which means only one of N_j is $2 \pmod 4$ and all others are odd, we see again from (2.2) that $\operatorname{sgn}(\zeta_{\mathbf{F}_1[G]}(-1)^N) = -1$.

Finally, when $4|N_1\cdots N_r$, it holds that $\operatorname{sgn}(\zeta_{\mathbf{F}_1[G]}(-1)^N) = 1$.

This completes the proof of Theorem. □

Bibliography

[C] H.B.G. Casimir: On the attraction between two perfectly conducting plates, *Proc. Kon. Nederland. Akad. Wetensch.* **B51** (1948), 793-795.

[CC] A. Connes and C. Consani: Schemes over $\mathbf{F}_1$ and zeta functions, arXiv 0903.2024v3 [math. AG] 9 Jul 2009.

[D] A. Deitmar: Schemes over $\mathbf{F}_1$, *Progress in Math.* **239** (2005), 87-100.

[DKK] A. Deitmar, S. Koyama and N. Kurokawa: Absolute zeta functions, *Proc. Japan Acad.* **84A** (2008), 138-142.

[Har] S. Haran: Non-additive geometry, *Compos. Math.* **143** (2007), 618-688.

[Haw] S.W. Hawking: Zeta function regularization of path integrals in curved space time, *Comm. Math. Phys.* **55** (1977), 133-148.

[K1] N. Kurokawa: Multiple zeta functions: an example, Zeta Functions in Geometry (Tokyo 1990), 219-226, *Adv. Stud. in Pure Math.* **21**, Kinokuniya, Tokyo, 1992.

[K2] N. Kurokawa: Zeta functions over $\mathbf{F}_1$, *Proc. Japan Acad.* **81A** (2005), 180-184.

[KKK] S. Kim, S. Koyama and N. Kurokawa: The Riemann hypothesis and functional equations for zeta functions over $\mathbf{F}_1$, *Proc. Japan Acad.* **85A** (2009), 75-80.

[KO] N.Kurokawa and H. Ochiai: A multivariable Euler products of Igusa type and its applications, *J. Number Theory* **129** (2009), 1919-1930.

[KOW] N.Kurokawa, H. Ochiai and M. Wakayama: Absolute derivations and zeta functions, *Doc. Math.* 2003, 565-584.

[KW1] N.Kurokawa and M. Wakayama: Casimir effects on Riemann surfaces, *Indag. Math. (N.S.)* **13** (2002), 63-75.

[KW2] N.Kurokawa and M. Wakayama: Absolute tensor products, *International Math. Research Notices* 2004, no. 5, 249-260.

[M] Yu. I. Manin: Lectures on zeta functions and motives (according to Deninger and Kurokawa), Columbia University Number Theory Seminar (New York, 1992), *Asterisque* **228** (1995), 121-163.

[S] C. Soulé: Les variete sur le corps á un element, *Moscow Math. J.* **4** (2004) , 217-244.

[T] J. Tits: Sur les analogues algebriques des groupes semi-simple complexes, Colloque d'algebre superieure, tenu a Bruxelles du 19 au 22 decembre 1956, Centre Belge de Recherches Math., Gauthier-Villar,Paris 1957, 261-289.

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