Absolute zeta functions

Shin-ya Koyama and Nobushige Kurokawa

Abstract. We first survey the absolute tensor product. In particular we find that zeta functions which are meromorphic of order r are expected to have Euler factors expressed in terms of the elliptic gamma function of order $r-1$. We give some observation and evidence for this phenomenon. Secondly, we generalize the absolute Weil type zeta function of Deitmar-Koyama-Kurokawa to noncommutative versions. We obtain the determinant expressions and give many examples.

1 Zeta Functions of Tensor Products over $\mathbf{F}_1$

We recall the construction of the absolute tensor product, which is called the Kurokawa tensor product by Manin [23]. We refer to [18] and [22] for details.

Let

$$
\begin{aligned}
Z_j(s) &= \prod_{\rho \in \mathbf{C}} (s - \rho)^{m_j(\rho)} \\
&= \exp\left(-\frac{\partial}{\partial w}\bigg|_{w=0} \sum_{\rho \in \mathbf{C}} \frac{m_j(\rho)}{(s-\rho)^w} \right)
\end{aligned}
$$

be "zeta functions" expressed as regularized product, where

$$
m_j \; : \; \mathbf{C} \to \mathbf{Z}
$$

denotes the multiplicity function for $j = 1, ..., r$. The absolute tensor product $(Z_1 \otimes \cdots \otimes Z_r)(s)$ is defined as

$$
(Z_1 \otimes \cdots \otimes Z_r)(s) = \prod_{\rho_1, \cdots, \rho_r \in \mathbf{C}} (s - (\rho_1 + \cdots + \rho_r))^{m(\rho_1, \cdots, \rho_r)}
$$

with

$$
m(\rho_1, \cdots, \rho_r) = m_1(\rho_1) \cdots m_r(\rho_r) \times \begin{cases} 1 & \operatorname{Im}(\rho_j) \geq 0, \quad (j = 1, ..., r) \\ (-1)^{r-1} & \operatorname{Im}(\rho_j) < 0, \quad (j = 1, ..., r) \\ 0 & \text{otherwise.} \end{cases}
$$

This definition originates from [15]. We refer to the excellent survey of Manin [23]. The notation of the regularized product is due to Deninger [9]. See [11] concerning the needed regularized products. The absolute tensor product was studied by Schröter [24] in the name of the "Kurokawa tensor product."

We are especially interested in the case of Hasse zeta functions $Z_j(s) = \zeta(s, A_j)$ for commutative rings $A_1, ..., A_r$ of finite type over $\mathbf{Z}$. We recall that the Hasse zeta function $\zeta(s, A)$ of a commutative ring A is defined to be

$$
\begin{aligned}
\zeta(s, A) &= \prod_{\mathbf{m}}(1 - N(\mathbf{m})^{-s})^{-1} \\
&= \exp\left(\sum_{\mathbf{m}}\sum_{k=1}^{\infty}\frac{1}{k}N(\mathbf{m})^{-ks}\right),
\end{aligned}
$$

where $\mathbf{m}$ runs over maximal ideals of A and $N(\mathbf{m}) = \#(A/\mathbf{m})$. This is also written as

$$
\zeta(s, A) = \exp\left(\sum_{p:\text{primes}}\sum_{m=1}^{\infty}\frac{|\text{Hom}_{\text{ring}}(A, \mathbf{F}_{p^m})|}{m}p^{-ms}\right).
$$

For simplicity we write

$$
\zeta(s, A_1 \otimes \cdots \otimes A_r) = \zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r).
$$

It holds that $\zeta(s, A_1 \otimes \cdots \otimes A_r)$ has the following additive structure on zeros and poles: if $\zeta(s_j, A_j) = 0$ or ∞ and $\text{Im}(s_j)$ $(j = 1, ..., r)$ have the same signature, then

$$
\zeta(s_1 + \cdots + s_r, A_1 \otimes \cdots \otimes A_r) = 0 \text{ or } \infty.
$$

Such an additive structure was crucial in the study of Hasse zeta functions of positive characteristic (congruence zeta functions) pursued by Grothendieck [12] and Deligne [6], where Euler products were important to restrict the region of zeros and poles for our reaching to the analogue of the Riemann Hypothesis.

We expect that our multiple zeta functions also have Euler products of the following form:

$$
\zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r) = \prod_{(\mathbf{m}_1, ..., \mathbf{m}_r)} H_{(\mathbf{m}_1, ..., \mathbf{m}_r)}(N(\mathbf{m}_1)^{-s}, ..., N(\mathbf{m}_1)^{-s})
$$

where $\mathbf{m}_i$ runs over the maximal ideals of A_i and $H_{(\mathbf{m}_1, ..., \mathbf{m}_r)}(T_1, ..., T_r)$ is a power series in $T_1, ..., T_r$ of the constant term 1 with a possible degeneration at $(\mathbf{m}_1, ..., \mathbf{m}_r)$, where $N(\mathbf{m}_i) = N(\mathbf{m}_j)$ for some $i \neq j$. More generally we expect that the multiple zeta function $Z_1(s) \otimes \cdots \otimes Z_r(s)$ has an Euler product

$$
Z_1(s) \otimes \cdots \otimes Z_r(s) = \prod_{(p_1, ..., p_r) \in P_1 \times \cdots \times P_r} H_{(p_1, ..., p_r)}(N(p_1)^{-s}, ..., N(p_r)^{-s})
$$

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when each zeta function $Z_j(s)$ has an Euler product

$$Z_j(s) = \prod_{p \in P_j} H_p^j(N(p)^{-s})$$

and a functional equation; here $H_p^j(T)$ is a power series in T and $H_{(p_1,...,p_r)}(T_1,...,T_r)$ is a power series in $(T_1,...,T_r)$ with a possible degeneration at $(p_1,...,p_r)$, where $N(p_i) = N(p_j)$ for some $i \neq j$.

In [18] we investigated the absolute tensor product $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ for primes p and q by using a signed double Poisson summation formula, where $\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}$. In other words we constructed a zeta function having zeros (or poles) at sums of poles of $\zeta(s, \mathbf{F}_p)$ and those of $\zeta(s, \mathbf{F}_q)$. We state the results as follows. Their proofs and some refinements were in [18] and [19]. For simplicity we use the notation $F(s) \cong G(s)$ for functions $F(s)$ and $G(s)$ to indicate that $F(s) = e^{Q(s)}G(s)$ for some polynomial $Q(s)$.

Theorem 1.1 *Let p and q be distinct prime numbers. Define the function $\zeta_{p,q}(s)$ in $\mathrm{Re}(s) > 0$ as follows:*

$$\zeta_{p,q}(s) := \exp\left(-\frac{i}{2} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log p}{\log q}\right)}{n} p^{-ns} - \frac{i}{2} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns} \right.$$
$$\left. -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} p^{-ns} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} q^{-ns} \right).$$

Then the function $\zeta_{p,q}(s)$ has the following properties:

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,q}(s)$ has an analytic continuation to all $s \in \mathbf{C}$ as a meromorphic function of order two.

(2) All zeros and poles of $\zeta_{p,q}(s)$ are simple and located at

$$s = 2\pi i \left(\frac{m}{\log p} + \frac{n}{\log q} \right),$$

where (m, n) is either a pair of nonnegative integers or a pair of negative integers. Indeed it gives a zero or pole according as they are nonnegative or negative.

(3) We have the identification

$$\zeta_{p,q}(s) \cong \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q).$$

(4) The function $\zeta_{p,q}(s)$ satisfies a functional equation:

$$\zeta_{p,q}(-s) = \zeta_{p,q}(s)^{-1}(pq)^{\frac{s}{2}}(1-p^{-s})(1-q^{-s})$$
$$\times \exp\left(\frac{i\log p\log q}{4\pi}s^2 - \frac{\pi i}{6}\left(\frac{\log q}{\log p} + \frac{\log p}{\log q} + 3\right)\right).$$

When $p = q$ the result is as follows:

Theorem 1.2 *Let*

$$\zeta_{p,p}(s) := \exp\left(\frac{i}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns} - \left(1 - \frac{i\log p}{2\pi}s\right)\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right)$$

in $\mathrm{Re}(s) > 0$. *Then the function* $\zeta_{p,p}(s)$ *has the following properties:*

(0) It converges absolutely in $\mathrm{Re}(s) > 0$.

(1) The function $\zeta_{p,p}(s)$ *has an analytic continuation to all* $s \in \mathbf{C}$ *as a meromorphic function of order two.*

(2) All zeros and poles of $\zeta_{p,p}(s)$ *are located at*

$$s = \frac{2\pi in}{\log p},$$

which gives a zero or pole of order $|n + 1|$, *according as n is a nonnegative or negative integer.*

(3) We have the identification

$$\zeta_{p,p}(s) \cong \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p).$$

(4) The function $\zeta_{p,p}(s)$ *satisfies a functional equation:*

$$\zeta_{p,p}(-s) = \zeta_{p,p}(s)^{-1}p^s(1-p^{-s})^2\exp\left(\frac{i(\log p)^2}{4\pi}s^2 - \frac{5\pi i}{6}\right).$$

The proof of Theorems 1.1 and 1.2 were done by constructing the signed double Poisson summation formula, and the theory of multiple sine functions developed in [17] are essential.

From our viewpoint, it is very interesting to see the nature of $\prod_{p,q}\zeta_{p,q}(s)$. Unfortunately, however, it does not converge even for sufficiently large $\mathrm{Re}(s)$. Our "α-version" $\zeta_{p,q}^{\alpha}(s)$ treated below remedies the situation. In passing we find the analyticity of the diagonal Euler product:

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Theorem 1.3 *Let*

$$Z(s) = \prod_p \zeta_{p,p}(s).$$

Then, $Z(s)$ is absolutely convergent in $\mathrm{Re}(s) > 1$, and it has an analytic continuation with singularities to $\mathrm{Re}(s) > 0$ with the natural boundary $\mathrm{Re}(s) = 0$.

Later in [19] we constructed "the double Riemann zeta function" $\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z})$ by establishing the signed double explicit formula which is stated in the following theorem. It generalizes the signed double Poisson summation formula used in the proof of Theorems 1.1 and 1.2.

For simplicity put $\xi(s) = \hat{\zeta}(s + \frac{1}{2}) = \hat{\zeta}(s + \frac{1}{2}, \mathbf{Z})$ for $\hat{\zeta}(s) = \Gamma_{\mathbf{R}}(s)\zeta(s)$ with $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. The functional equation of $\zeta(s)$ is written as $\xi(s) = \xi(-s)$. We recall that nontrivial zeros of $\zeta(s)$ are zeros in the strip $|\mathrm{Re}(s - \frac{1}{2})| < 1/2$. We denote by $\frac{1}{2} + i\gamma$ such a zero, where γ is a complex number in $-1/2 < \mathrm{Im}(\gamma) < 1/2$.

Hereafter let $h(t)$ be an odd regular function in $|\mathrm{Re}(t)| < 1$ satisfying $h(t) = O(|t|^{-3})$ as $|t| \to \infty$. We put $H_\alpha(t) := h(2\alpha + it)$ and

$$\widetilde{H}(u) := \int_{-\infty}^{\infty} H(t)e^{itu}dt.$$

Theorem 1.4 *Let $1/2 < \alpha < 1$. We have*

$$\sum_{\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)>0} H_0(\gamma_1 + \gamma_2) = \sum_{\substack{p,q \\ p\neq q}} \mathcal{H}_{p,q}^\alpha + \sum_p \mathcal{H}_{p,p}^\alpha + \sum_p \mathcal{H}_{p,\infty}^\alpha + \mathcal{H}_{\infty,\infty}^\alpha + \mathcal{H}_0^\alpha,$$

where the sum in the left hand side is taken over pairs $(\frac{1}{2} + i\gamma_1, \frac{1}{2} + i\gamma_2)$ of nontrivial zeros of the Riemann zeta function, the sum in the right hand side is taken over pairs of distinct primes p, q or primes p, and we define for pairs of distinct primes p, q as

$$\mathcal{H}_{p,q}^\alpha = \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(p^m q^n)} \frac{1}{p^{m(\alpha+\frac{1}{2})}q^{n(\alpha+\frac{1}{2})}} \left(\widetilde{H}_0(-m\log p) + \widetilde{H}_0(-n\log q)\right) \qquad (1.1)$$

$$+ \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p \log q}{\log(\frac{p^m}{q^n})} \frac{1}{p^{m(\alpha+\frac{1}{2})}q^{n(\alpha+\frac{1}{2})}} \left(\widetilde{H}_\alpha(-m\log p) - \widetilde{H}_\alpha(-n\log q)\right), \quad (1.2)$$

and for a prime p,

$$\mathcal{H}_{p,p}^\alpha = \frac{i}{4\pi^2} \sum_{m,n} \frac{\log p}{(m+n)p^{(m+n)(\alpha+\frac{1}{2})}} \left(\widetilde{H}_0(-m\log p) + \widetilde{H}_0(-n\log p)\right) \qquad (1.3)$$

$$+ \frac{i}{4\pi^2} \sum_{m\neq n} \frac{\log p}{(m-n)p^{(m+n)(\alpha+\frac{1}{2})}} \left(\widetilde{H}_\alpha(-m\log p) - \widetilde{H}_\alpha(-n\log p)\right) \qquad (1.4)$$

$$+ \frac{1}{4\pi^2}(\log p)^2 \sum_{m=1}^{\infty} p^{-2m(\alpha+\frac{1}{2})}\widetilde{tH_\alpha(t)}(-m\log p), \qquad (1.5)$$

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$$
\mathcal{H}^{\alpha}_{p,\infty} = -\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\log p}{p^{m(\alpha+\frac{1}{2})}} \int_{-\infty}^{\infty} p^{-imt} H_{\alpha}(t) \int_0^t p^{imt'} \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it'\right) dt' dt \qquad (1.6)
$$

$$
-\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\log p}{p^{m(\alpha+\frac{1}{2})}} \int_{-\infty}^{\infty} H_0(t) \int_0^t \mathrm{Re}\left(p^{im(t-t')} \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it'\right)\right) dt' dt, \quad (1.7)
$$

$$
\mathcal{H}^{\alpha}_{\infty,\infty} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_{\alpha}(t) \int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right) \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+i(t-t_1)\right) dt_1 dt \qquad (1.8)
$$

$$
+\frac{1}{4\pi^2} \int_{-\infty}^{\infty} H_0(t) \int_0^t \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}+it_1\right) \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}\left(\alpha+\frac{1}{2}-i(t-t_1)\right) dt_1 dt, \qquad (1.9)
$$

and

$$
\mathcal{H}^{\alpha}_0 = -\frac{\alpha}{\pi} \int_0^{\pi} \sum_{\mathrm{Re}(\gamma_1)>0} h(i\gamma_1 + \alpha e^{i\theta}) \frac{\xi'}{\xi}(\alpha e^{i\theta}) e^{i\theta} d\theta \qquad (1.10)
$$

$$
-\frac{\alpha^2}{4\pi^2} \int_0^{\pi}\int_0^{\pi} h(\alpha e^{i\theta_1} + \alpha e^{i\theta_2}) \frac{\xi'}{\xi}(\alpha e^{i\theta_1}) \frac{\xi'}{\xi}(\alpha e^{i\theta_2}) e^{i(\theta_1+\theta_2)} d\theta_1 d\theta_2, \qquad (1.11)
$$

where $m, n \in \mathbf{Z}$, $m, n \geq 1$.

Notice that only pairs of zeros in the upper (or lower) half plane are counted in the left hand side of Theorem 1.4. The method of Cramér [5] is important in the proof.

For defining the (p, q)-Euler factors, we put

$$
h(t) = \frac{1}{(t+s)^2} - \frac{1}{(t-s)^2} \qquad (1.12)
$$

in Theorem 1.4. We denote by p, q any (finite or infinite) places. We call $\zeta^{\alpha}_{p,q}(s)$ a (p, q)-Euler factor of the double Riemann zeta function $\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z})$ if and only if it holds that

$$
\zeta^{\alpha}_{p,q}(s+1) = \exp\left(\iint \mathcal{H}^{\alpha}_{p,q}(s) ds ds\right).
$$

We also denote the remainder factor $\zeta^{\alpha}_0(s)$ by

$$
\zeta^{\alpha}_0(s+1) = \exp\left(\iint \mathcal{H}^{\alpha}_0(s) ds ds\right).
$$

Note that these definitions implies some ambiguity emerged from the integral constants, that is the factor $\exp(Q(s))$ with $Q(s)$ a polynomial with $\deg(Q) \leq 2$. The double Riemann zeta function $\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z})$ is expressed by an Euler product over the pairs of places (p, q). In the following theorem we denote the dilogarithm of order r by $\mathrm{Li}_r(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^r}$ $(r = 1, 2, 3, ...)$.

Theorem 1.5 *The (p,q)-Euler factors of the double Riemann zeta function $\zeta(s,\mathbf{Z})\otimes\zeta(s,\mathbf{Z})$ are described as follows:*

(1) For distinct prime numbers p and q, we put

$$
\zeta_{p,q}^{\alpha}(s) \;=\; \exp\left(\frac{1}{\pi i}\sum_{m,n} \frac{(\log p)(\log q)}{(m\log p)^2-(n\log q)^2} \right.
$$
$$
\left(\frac{\cosh(m\alpha\log p)}{q^{n(\alpha+\frac12)}} p^{-m(s-\frac12)} + \frac{n\log q}{m\log p}\frac{\sinh(m\alpha\log p)}{q^{n(\alpha+\frac12)}} p^{-m(s-\frac12)} \right.
$$
$$
\left.\left. - \frac{m\log p}{n\log q}\frac{\sinh(n\alpha\log q)}{p^{m(\alpha+\frac12)}} q^{-n(s-\frac12)} - \frac{\cosh(n\alpha\log q)}{p^{m(\alpha+\frac12)}} q^{-n(s-\frac12)} \right)\right)
$$

in $\mathrm{Re}(s)>\alpha+\frac12$, where the sum is taken over all pairs of all positive integers m and n. Then it is a (p,q)-Euler factor of the double Riemann zeta function, and it has an analytic continuation to the entire plane.

(2) For a prime number p, a (p,p)-Euler factor is given as follows in $\mathrm{Re}(s)>\alpha+\frac12$:

$$
\zeta_{p,p}^{\alpha}(s) \;=\; \exp\left(\frac{2}{\pi i}\sum_{m\neq n} \frac{p^{-m(s-\frac12)-n(\alpha+\frac12)}}{m^2-n^2}\left(\cosh(m\alpha\log p)+\frac{n}{m}\sinh(m\alpha\log p)\right) \right. \quad (1.13)
$$
$$
\left. +\frac{1}{2\pi i}\left((\log p)(s-1-2\alpha)\log(1-p^{-s})-\mathrm{Li}_2\left(p^{-s}\right)+\mathrm{Li}_2\left(p^{-s-2\alpha}\right)\right)\right). (1.14)
$$

It has an analytic continuation to the entire plane.

(3) The (p,∞)-factor $\zeta_{p,\infty}^{\alpha}(s)$ of the double Riemann zeta function has an analytic continuation to the entire plane, and moreover $\prod_p \zeta_{p,\infty}^{\alpha}(s)$ has an analytic continuation to the entire plane with possible singularities at $s=\frac12-2k\pm\alpha$ $(k\geq 0)$, $1-2k$ $(k\geq 0)$, $-2k$ $(k\geq 1)$, $\rho-2k$ $(k\geq 0)$, $\frac12+\rho\pm\alpha$, $\frac32\pm\alpha$ with ρ any nontrivial zero of $\zeta(s)$.

(4) The (∞,∞)-factor $\zeta_{\infty,\infty}^{\alpha}(s)$ of the double Riemann zeta function is analytic with possible singularities at $s=-2n$, $-2n+\alpha+\frac12$ with $n=0,1,2,...$

Theorem 1.6 *The remaining factor $\zeta_0^{\alpha}(s)$ of the double Riemann zeta function is an analytic function on $\mathbf{C}$ with possible singularities at $s=\rho+\frac12+\mathrm{sgn}(\mathrm{Im}(\rho))\alpha e^{i\theta}$ with $0\leq\theta\leq\pi$ for any nontrivial zero ρ of $\zeta(s)$ and at s belonging to $|s-1|\leq 2\alpha$.*

In the next theorem we use the half Riemann zeta function $\zeta_+(s)$ studied in [11] and the multiple gamma function $\Gamma_r(s)$ of Barnes [4].

Theorem 1.7 *The Euler product for the double Riemann zeta function*

$$\zeta(s, \mathbf{Z}) \otimes \zeta(s, \mathbf{Z}) \cong \left(\prod_{p,q} \zeta_{p,q}^{\alpha}(s) \right) \zeta_0^{\alpha}(s) \left(\frac{\prod_{m=1}^{\infty} \zeta_+(s + 2m)}{\zeta_+(s - 1)} \right)^2 \Gamma_2 \left(\frac{s}{2} \right)^{-1} \Gamma_1(s)^2 s(s - 2)$$

is absolutely convergent in $\mathrm{Re}(s) > \alpha + \frac{3}{2}$, *where* (p, q) *runs through pairs of all (finite or infinite) places. It has an analytic continuation (with singularities) to the entire plane and satisfies a functional equation between* s *and* $2 - s$.

Later Akatsuka [1] successfully eliminated the parameter α from the double Riemann zeta function. He obtained the (p, q)-Euler factor

$$\zeta_{p,q}(s) = \exp \left(\frac{1}{\pi i} \sum_p \sum_{m=1}^{\infty} \sum_q \sum_{\substack{n=1 \\ q^n \neq p^m}}^{\infty} \frac{p^{-m(s-1)} q^{-n} \log p}{n(m \log p - n \log q)} \right.$$

$$\left. - \frac{1}{\pi i} \sum_p \sum_{m=1}^{\infty} \sum_q \sum_{n=1}^{\infty} \frac{p^{-ms} q^{-n} \log p}{n(m \log p + n \log q)} \right),$$

and proved that the double Euler product

$$\zeta^{\otimes 2}(s) = \prod_{p,q} \zeta_{p,q}(s)$$

satisfies the similar property as in Theorem 1.5.

For $r \geq 0$ and $x, q_j \in \mathbf{C}$ $(j = 1, ..., r)$ we define the *elliptic gamma function* of order r by

$$G_r(x; q_1, ..., q_r) := \prod_{n_1, ..., n_r \geq 0} (1 - x q_1^{n_1} \cdots q_r^{n_r}). \tag{1.15}$$

We conventionally put $G_0(x) = 1 - x$. The product in (1.15) is absolutely convergent when $|q_j| < 1$ for $j = 1, ..., r$.

The function (1.15) was originally dealt with by Appell [2]. He actually considered

$$O_q(x; \omega_1, ..., \omega_r) = \prod_{n_1, ..., n_r \geq 0} (1 - q^{n_1 \omega_1 + \cdots + n_r \omega_r + x})$$

$$= G_r(q^x; q^{\omega_1}, ..., q^{\omega_r})$$

with $\mathrm{Re}(\omega_j) > 0$ $(j = 1, ..., r)$ and $0 < q < 1$.

The elliptic gamma function (1.15) has another expression (Kurokawa-Wakayama [21, Proposition 2.1]).

$$G_r(x; q_1, ..., q_r) = \exp\left(-\sum_{m=1}^{\infty} \frac{x^m}{m(1 - q_1^m)\cdots(1 - q_r^m)} \right). \tag{1.16}$$

The sum in (1.16) is absolutely convergent for suitable $q_j \in \mathbf{C}$ outside of $|q_j| < 1$. For example, it is valid for $q_j = e^{2\pi i \alpha_j}$ with $\alpha_j \in \mathbf{R}$
Z such that either $\alpha_j \in (\overline{\mathbf{Q}} \setminus \mathbf{Q}) \cap \mathbf{R}$ (Roth) or $\alpha_j = \log p / \log q$ with p and q distinct primes (Baker).

In [21, Theorem 1.1] Kurokawa and Wakayama proved the following theorem.

Theorem 1.8 *Let $p_1, ..., p_r$ be distinct prime numbers. Then for $\mathrm{Re}(s) > 0$,*

$$\zeta(s, \mathbf{F}_{p_1}) \otimes \cdots \otimes \zeta(s, \mathbf{F}_{p_r})$$
$$= \left(G_{r-1}\left(p_1^{-s}; \exp\left(\frac{2\pi i \log p_1}{\log p_2} \right), \cdots, \exp\left(\frac{2\pi i \log p_1}{\log p_r} \right) \right) \times \cdots \right.$$
$$\left. \times G_{r-1}\left(p_r^{-s}; \exp\left(\frac{2\pi i \log p_r}{\log p_1} \right), \cdots, \exp\left(\frac{2\pi i \log p_r}{\log p_{r-1}} \right) \right) \right)^{(-1)^r}.$$

Here, G_{r-1} is considered under the expression (1.16) and the convergence comes from a delicate transcendency result of Baker [3, Theorem 3.1]

In particular, when $r = 2$, we have for distinct prime numbers p and q

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) = G_1\left(p^{-s}; \exp\left(\frac{2\pi i \log p}{\log q} \right) \right) G_1\left(q^{-s}; \exp\left(\frac{2\pi i \log q}{\log p} \right) \right).$$

As $\zeta(s, \mathbf{F}_{p_1}) \otimes \cdots \otimes \zeta(s, \mathbf{F}_{p_r})$ is a meromorphic function of order r, we reach the following expectation.

Expectation.

Zeta functions which are meromorphic of order r have Euler products with Euler factors being expressed in terms of G_{r-1}.

Example. 1.1 (Dedekind zeta functions ($r = 1$))

Let K be a number field. The Dedekind zeta function of K is
$$\zeta_K(s) = \prod_{p}(1 - N(p)^{-s})^{-1} = \prod_{p} G_0(N(p))^{-1},$$

where p runs through all maximal ideals of O_K and $N(p) = |O_K/(p)|$.

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Example. 1.2 (Selberg zeta functions for $SL(2, \mathbf{R})$ $(r = 2)$)

Let $\Gamma \subset SL(2, \mathbf{R})$ be a discrete subgroup. The Selberg zeta function of Γ is

$$Z_\Gamma(s) = \prod_{n=0}^{\infty} \prod_{p} (1 - N(p)^{-s-n}) = \prod_{p} G_1(N(p)^{-s}; N(p)^{-1}),$$

where p runs through all primitive hyperbolic conjugacy classes of Γ, and $N(p)$ is the larger square of the eigenvalues of p.

Example. 1.3 (Selberg zeta functions for $SL(2, \mathbf{C})$ $(r = 3)$)

Let $\Gamma \subset SL(2, \mathbf{C})$ be a discrete subgroup. The Selberg zeta function of Γ is

$$Z_\Gamma(s) = \prod_{p} \prod_{\substack{k \geq 0, \ l \geq 0 \\ k \equiv l \pmod{\nu_p}}} (1 - a(p)^{-2k} \overline{a(p)}^{-2l} N(p)^{-s}),$$

where p runs through certain hyperbolic conjugacy classes of Γ, the eigenvalues of p are denoted by $a(p)$ and $\overline{a(p)}$ with $|a(p)| > 1$, and $N(p) = |a(p)|^2$. The symbol ν_p is defined as $\nu_p = |(\Gamma_p)_{\mathrm{tor}}|$, the order of the torsion of the centralizer of p in Γ. For such hyperbolic classes p that $\nu_p = 1$, the Euler factor has the form

$$\prod_{l=0}^{\infty} \prod_{k=0}^{\infty} (1 - a(p)^{-2k} \overline{a(p)}^{-2l} N(p)^{-s}) = G_2(N(p)^{-s}; a(p)^{-2}, \overline{a(p)}^{-2}).$$

2 Absolute Weil Zeta Functions

For a group G we denote $\mathbf{F}_1[G] = G \cup \{0\}$, where 0 is an element satisfying $0 \cdot g = g \cdot 0 = 0$ for any $x \in G$. For a positive integer $m \geq 1$, we denote

$$\begin{aligned} \mathbf{F}_{1^m} &= \mathbf{F}_1[\boldsymbol{\mu}_m] \\ &= \{0\} \cup \boldsymbol{\mu}_m \\ &= \{0\} \cup \{z \in \mathbf{C} \mid z^m = 1\}. \end{aligned}$$

We define an $\mathbf{F}_1$-algebra as a multiplicative monoid with 0.

Let $1 \leq r \in \mathbf{Z}$. For a $\mathbf{Z}$-algebra A, we recall that the noncommutative Hasse zeta function of A as

$$\zeta_{\mathbf{Z}}^r(s, A) = \exp\left(\sum_{p: \text{ prime}} \sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(A, M_r(\mathbf{F}_{p^m}))|}{m} p^{-ms} \right), \tag{2.1}$$

where $M_r(\mathbf{F}_{p^m})$ is the ring of matrices over $\mathbf{F}_{p^m}$ of size r, and Hom is the set of ring homomorphisms.

When $r = 1$ and A is a commutative finitely generated $\mathbf{Z}$-algebra, the definition (2.1) coincides with the usual Hasse zeta function. Namely, we have an expression

$$\zeta_{\mathbf{Z}}^1(s, A) = \prod_{\substack{M \subset A \\ \text{maximal ideal}}} (1 - N(M)^{-s})^{-1}$$

with $N(M) = |A/M|$. When $r \geq 2$ and A is a commutative finitely generated $\mathbf{Z}$-algebra, Kurokawa [14] and Fukaya [10] study some analogous versions of (2.1).

When A is an $\mathbf{F}_1$-algebra, we analogously define

$$\zeta_{\mathbf{F}_1}^r(s, A) = \exp\left(\sum_{m=1}^{\infty} \frac{|\text{Hom}(A, M_r(\mathbf{F}_{1^m}))|}{m} e^{-ms}\right), \tag{2.2}$$

where Hom is the set of monoid homomorphisms.

When $r = 1$ and A is a finitely generated abelian monoid, which is an $\mathbf{F}_1$-algebra, (2.2) was studied in [8] and [13]. The explicit form and the functional equation of $\zeta_{\mathbf{F}_1}(s, A)$ are given by the following proposition and theorem.

Proposition 2.1 ([8],Proposition 2.1) *Assume $A = \mathbf{F}_1[G] = G \cup \{0\}$ with G a finitely generated multiplicative abelian group of rank r. Put*

$$G \cong \mathbf{Z}^r \times \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1 | n_2 | \cdots | n_k$. Then

$$\zeta_{\mathbf{F}_1}(s, A) = \begin{cases} \displaystyle\prod_{\underline{d}|\underline{n}} \left(1 - e^{-|\underline{d}|s}\right)^{-\varphi(\underline{d})d_1^{k-2}\cdots d_k^{-1}} & for \quad r = 0, \\[2em] \displaystyle\prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(e^{-|\underline{d}|s})\varphi(\underline{d})d_1^{r+k-1}\cdots d_k^{r-1}}{(1-e^{-|\underline{d}|s})^r}\right) & for \quad r \geq 1, \end{cases}$$

where the notation $\underline{d}|\underline{n}$ means that the product is over all tuples

$$\underline{d} = (d_1, \ldots, d_k) \in \mathbf{N}^k$$

such that $d_1|n_1$, $d_2|\frac{n_2}{d_1}$,..., $d_k|\frac{n_k}{d_1\cdots d_{k-1}}$. Further, we put

$$|\underline{d}| = d_1 \cdots d_k$$

and

$$\varphi(\underline{d}) = \varphi(d_1) \cdots \varphi(d_k).$$

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Theorem 2.1 ([13],Theorem 2)   (i) *Assume* $A = \mathbf{F}_1[G] = G \cup \{0\}$ *with* G *a finite abelian group of order* n. *Put*

$$G \cong \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1|n_2| \cdots |n_k$ *and* $n = n_1 \cdots n_k$. *Then*

$$\zeta_{\mathbf{F}_1}(s, A) = \det\left(1 - \Phi_A e^{-s}\right)^{-1/n} \tag{2.3}$$

and the following functional equation holds:

$$\zeta_{\mathbf{F}_1}(-s, A) = w(A)e^{-ns}\zeta_{\mathbf{F}_1}(s, A),$$

where $w(A)$ *is a complex number of modulus 1 satisfying*

$$w(A) = (-1)^n \det(\Phi_A)^{\frac{1}{n}},$$

and where we define the absolute Frobenius operator Φ_A *on the group*

$$G^{(2)} = \boldsymbol{\mu}_{n_1^2} \times \boldsymbol{\mu}_{n_2^2} \times \cdots \times \boldsymbol{\mu}_{n_k^2}$$

as

$$\Phi_A(\alpha_1, ..., \alpha_k) = (\alpha_1^{n_1+1}, ..., \alpha_k^{n_k+1}). \tag{2.4}$$

(ii) *Assume* $A = \mathbf{F}_1[G] = G \cup \{0\}$ *with* G *a finitely generated free abelian group of rank 1. Put*

$$G \cong \mathbf{Z} \times \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1|n_2| \cdots |n_k$. *Then the following functional equations hold:*

$$\zeta_{\mathbf{F}_1}(-s, A) = \zeta_{\mathbf{F}_1}(s, A)^{-1} \prod_{\underline{d}|\underline{n}} e^{-\varphi(\underline{d})d_1^{k-1}\cdots d_k}.$$

(iii) *If* $A = \mathbf{F}_1[G] = G \cup \{0\}$ *with* G *is a finitely generated abelian group of rank* $r \geq 2$, *the following functional equation holds:*

$$\zeta_{\mathbf{F}_1}(-s, A) = \zeta_{\mathbf{F}_1}(s, A)^{(-1)^r}.$$

Above all the determinant expression (2.3) was crutial. By this we proved an analogue of the Riemann hypothesis as well as a tensor structure of the zeta functions ([13]). We generalize (2.3) to $r \geq 2$ in the following theorem.

Theorem 2.2 *Assume all prime divisors of n are bigger than r. In other words, we assume that*

$$p \mid n \qquad \Longrightarrow \qquad p > r.$$

Let $A = \mathbf{F}_1[G] = G \cup \{0\}$ with G a finite abelian group of order n. Then $\zeta_{\mathbf{F}_1}^r(s, A)$ has an Euler product

$$\zeta_{\mathbf{F}_1}^r(s, A) = \prod_{d \mid n} \left(1 - e^{-ds}\right)^{-\frac{1}{d}\left(d^r - \sum_{d' \mid d} \varphi(d')^r\right)} \tag{2.5}$$

and a determinant expression

$$\zeta_{\mathbf{F}_1}^r(s, A) = \det\left(1 - \Phi_A^{\otimes r} e^{-s}\right)^{-1/n^r}, \tag{2.6}$$

where Φ_A is defined by (2.4) which is a square matrix of size n^2, and $\Phi_A^{\otimes r}$ denotes the Kronecker tensor power, which is a square matrix of size n^{2r}.

    Proof. Put

$$G \cong \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1 | n_2 | \cdots | n_k$ and $n = n_1 \cdots n_k$. We first claim

$$\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m})) = (n_1, m)^r \cdots (n_k, m)^r. \tag{2.7}$$

Then (2.5) follows from a direct calculation. For proving (2.6), it suffices to prove that

$$\mathrm{Hom}(\boldsymbol{\mu}_l, M_r(\mathbf{F}_{1^m})) = (l, m)^r,$$

by our considering the image of a generator ζ_l of each $\boldsymbol{\mu}_l$ ($l = n_1, ..., n_k$).

    Since the image of ζ_l has to be invertible, it holds that

$$|\mathrm{Hom}(\boldsymbol{\mu}_l, M_r(\mathbf{F}_{1^m}))| = |\mathrm{Hom}(\boldsymbol{\mu}_l, GL_r(\mathbf{F}_{1^m}))|.$$

Here $GL_r(\mathbf{F}_{1^m})$ is the group of invertible matrices in the form of a permutation matrix with the entry 1 replaced by any element in $\boldsymbol{\mu}_m$.

    Denote by α the image of ζ_l by a homomorphism. By the assumption, α is a diagonal matrix with diagonal entries in $\boldsymbol{\mu}_m$, because the order of α must be equal to l, not divisible by any positive integer less than l. Any element $\alpha \in \begin{pmatrix} \boldsymbol{\mu}_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\mu}_m \end{pmatrix}$ with $\alpha^l = 1$ can be an image. Thus

$$|\mathrm{Hom}(\boldsymbol{\mu}_l, M_2(\mathbf{F}_{1^m}))| = \left| \left\{ \alpha \in \begin{pmatrix} \boldsymbol{\mu}_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\mu}_m \end{pmatrix} : \alpha^l = 1 \right\} \right| = (l, m)^r.$$

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This proves (2.7). Then we have

$$\zeta^r_{\mathbf{F}_1}(s, A) = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))|}{m} e^{-ms}\right)$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{(n_1, m)^r \cdots (n_k, m)^r}{m} e^{-ms}\right).$$

For proving (2.6), it suffices to show that

$$\sum_{m=1}^{\infty} \frac{n^r (n_1, m)^r \cdots (n_k, m)^r}{m} e^{-ms} = \sum_{m=1}^{\infty} \frac{\mathrm{tr}((\Phi_A^{\otimes r})^m)}{m} e^{-ms} \qquad (2.8)$$

by taking the logarithm of (2.6). From a general identity $\mathrm{tr}((A \otimes B)^m) = \mathrm{tr}(A^m \otimes B^m) = \mathrm{tr}(A^m)\mathrm{tr}(B^m)$, we have

$$\mathrm{tr}((\Phi_A^{\otimes r})^m) = \mathrm{tr}((\Phi_A^m)^r).$$

On the other hand, it holds by [13, Lemma 1] that

$$\mathrm{tr}(\Phi_A^m) = n\,|\mathrm{Hom}(A, \boldsymbol{\mu}_m)| = n(n_1, m) \cdots (n_k, m).$$

These identities lead to (2.8). ∎

Example. 2.1 For any prime $p > r$, it holds that

$$\zeta^r_{\mathbf{F}_1}(s, \mathbf{F}_{1^p}) = (1 - e^{-s})^{-1}(1 - e^{-ps})^{-\frac{p^r - 1}{p}}.$$

By this theorem, it is easily shown that $\zeta^r_{\mathbf{F}_1}(s, A)$ satisfies an analog of the Riemann hypothesis and the tensor structure concerning zeros and poles. The proofs are the same as in [13, Theorem 3]. The functional equation is also obtained as follows.

Corollary 1 *The zeta function in the preceding theorem satisfies the funcitonal equation*

$$\zeta^r_{\mathbf{F}_1}(-s, A) = \zeta^r_{\mathbf{F}_1}(s, A)(-1)^{n^r} \det(\Phi_A)^{rn^{r-2}} e^{-n^r s}.$$

Proof. In the previous paper [13, Proposition 1] we obtained the functional equation of the zeta function $\zeta_\sigma(s) = \det(1 - \sigma e^{-s})^{-1}$ of a finite permutation matrix σ over N elements as

$$\zeta_\sigma(-s) = (-1)^N \det(\sigma) e^{-Ns} \zeta_\sigma(s).$$

Identifying $\Phi_A^{\otimes r}$ with a permutation matrix over n^{2r} elements, we compute by the theorem that

$$\zeta^r_{\mathbf{F}_1}(-s, A) = \zeta_{\Phi_A^{\otimes r}}(-s)^{\frac{1}{n^r}}$$

$$= \left(\zeta_{\Phi_A^{\otimes r}}(s)(-1)^{n^{2r}} \det(\Phi_A^{\otimes r}) e^{-n^{2r} s}\right)^{\frac{1}{n^r}}$$

$$= \left(\zeta_{\Phi_A^{\otimes r}}(s)(-1)^{n^{2r}} \det(\Phi_A)^{rn^{2(r-1)}} e^{-n^{2r} s}\right)^{\frac{1}{n^r}}$$

$$= \zeta^r_{\mathbf{F}_1}(s, A)^{\frac{1}{n^r}} (-1)^{n^r} \det(\Phi_A)^{rn^{r-2}} e^{-n^r s}. \blacksquare$$

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Investigating the cases when n has small prime divisors is an interesting problem. It seems that the situation is much more complicated. We give some examples below.

Example. 2.2 When $A = \mathbf{F}_1[\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2]$, we have

$$\zeta_{\mathbf{F}_1}^2(s, A) = (1 - e^{-s})^{-25} \exp\left(\frac{10 - 36e^{-s} + 9e^{-2s} + 18e^{-3s}}{(1 - e^{-s})^2}\right).$$

Proof. The result follows from

$$|\mathrm{Hom}(A, M_2(\mathbf{F}_{1^m}))| = \begin{cases} (m + 5)^2 & m \geq 3 \\ (m + 2)^2 & m = 1, 2 \end{cases}.$$

∎

Example. 2.3 When $A = \mathbf{F}_{1^3}$, we have

$$\zeta_{\mathbf{F}_1}^3(s, A) = (1 - e^{-s})^{-1}(1 - e^{-3s})^{-\frac{20}{3}} \exp\left(3\frac{e^{-3s}(1 + 2e^{-3s})}{(1 - e^{-3s})^2}\right).$$

Proof. The result follows from

$$|\mathrm{Hom}(A, M_3(\mathbf{F}_{1^m}))| = \begin{cases} 1 & 3 \nmid m \\ 27 + 2^2 & 3 \mid m \end{cases}.$$

∎

Example. 2.4 When $A = \mathbf{F}_{1^2}$, we have

$$\zeta_{\mathbf{F}_1}^2(s, A) = (1 - e^{-s})^{-5} \exp\left(\frac{-4 + 3e^{-s} + 3e^{-2s}}{2(e^{-s} - 1)}\right).$$

Proof. The result follows from

$$|\mathrm{Hom}(A, M_2(\mathbf{F}_{1^m}))| = \begin{cases} m + 5 & m \geq 3 \\ m + 2 & m = 1, 2 \end{cases}.$$

∎

Example. 2.5 For $A = \mathbf{F}_{1^2}$, we have

$$\zeta_{\mathbf{F}_1}^r(s, A) = \exp\left(\sum_{l=1}^{\infty} \frac{r!}{(2l-1)!} \sum_{k=0}^{[r/2]} \frac{e^{-2ls}}{2^k(r - 2k)!} \left(\frac{(2l)^k}{2l - 1}e^s + \frac{(2l + 1)^k}{(2l)^2}2^{2(l-k)}\right)\right)$$

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Proof. The result follows from

$$|\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))| = \begin{cases} \sum_{k=0}^{[r/2]} \dfrac{r!(m+1)^k}{2^k(r-2k)!} & 2 \nmid m \\ \sum_{k=0}^{[r/2]} \dfrac{r!(m+1)^k}{2^k(r-2k)!} 2^{m-2k} & 2 \mid m \end{cases}.$$

∎

In what follows we give some examples for the case when A is infinite. We define the Euler polynomials $g_r(T) \in \mathbb{Z}[T]$ $(r = 1, 2, 3, ...)$ by

$$g_1(T) = T \quad \text{and} \quad g_{r+1}(T) = \sum_{k=1}^{r} \binom{r}{k-1}(T-1)^{r-k} g_k(T). \tag{2.9}$$

For example,

$$\begin{aligned} g_1(T) &= T, \\ g_2(T) &= T, \\ g_3(T) &= T^2 + T, \\ g_4(T) &= T^3 + 4T^2 + T. \end{aligned}$$

In [8, Lemma 2.2], we proved the following lemma.

Lemma 2.1 *For $r = 1, 2, 3, ...$, we have*

$$\sum_{\nu=1}^{\infty} \nu^{r-1} T^\nu = \frac{g_r(T)}{(1-T)^r},$$

where $g_r(T) \in \mathbb{Z}[T]$ is defined by (2.9).

By using this lemma, we prove the following.

Example. 2.6 When $A = \mathbf{F}_1[T^\pm]$, the explicit form of $\zeta^r_{\mathbf{F}_1}(s, A)$ is given by

$$\zeta^r_{\mathbf{F}_1}(s, A) = \exp\left(\frac{g_r(e^{-s})r!}{(1-e^{-s})^r}\right).$$

It satisfies the functional equation

$$\zeta^r_{\mathbf{F}_1}(-s, A) = \zeta^r_{\mathbf{F}_1}(s, A)^{(-1)^r}.$$

Proof. Since an element in $\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))$ is determined by the image of T, which must be invertible. Thus

$$|\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))| = |GL_r(\mathbf{F}_{1^m})| = m^r r!.$$

Then we compute

$$\zeta_{\mathbf{F}_1}^r(s, A) = \exp\left(\sum_{m=1}^{\infty} \frac{m^r r!}{m} e^{-ms}\right)$$

$$= \exp\left(\sum_{m=1}^{\infty} m^{r-1} e^{-ms}\right)^{r!}$$

$$= \exp\left(\frac{g_r(e^{-s})r!}{(1 - e^{-s})^r}\right).$$

∎

Example. 2.7 When $A = \mathbf{F}_1[T]$, the explicit form of $\zeta_{\mathbf{F}_1}^r(s, A)$ is given by

$$\zeta_{\mathbf{F}_1}^r(s, A) == (1 - e^{-s})^{-1} \exp\left(\sum_{k=1}^{r} \binom{r}{k} r^k \frac{g_k(e^{-s})}{(1 - e^{-s})^k}\right).$$

Proof. The element in $\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))$ is determined by the image of T, which must belong to $M_r(\mathbf{F}_{1^m}) = \mathrm{End}(\mathbf{F}_{1^m}^r)$. Thus

$$|\mathrm{Hom}(A, M_r(\mathbf{F}_{1^m}))| = |\mathrm{End}(\mathbf{F}_{1^m}^r)| = (rm + 1)^r.$$

Then

$$\zeta_{\mathbf{F}_1}^r(s, A) = \exp\left(\sum_{m=1}^{\infty} \frac{(rm + 1)^r}{m} e^{-ms}\right)$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=0}^{r} \binom{r}{k} (rm)^k e^{-ms}\right)$$

$$= \exp\left(\sum_{m=1}^{\infty} \left(\frac{1}{m} + \sum_{k=1}^{r} \binom{r}{k} r^k m^{k-1}\right) e^{-ms}\right)$$

$$= (1 - e^{-s})^{-1} \exp\left(\sum_{k=1}^{r} \binom{r}{k} r^k \sum_{m=1}^{\infty} m^{k-1} e^{-ms}\right)$$

$$= (1 - e^{-s})^{-1} \exp\left(\sum_{k=1}^{r} \binom{r}{k} r^k \frac{g_k(e^{-s})}{(1 - e^{-s})^k}\right)$$

∎

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References

[1] H. Akatsuka: The double explicit formula and the double Riemann zeta function, (preprint) 2007.

[2] P. Appell: Sur une classe de fonctions analogues aux fonctions Eulériennes, Math. Ann. **19** (1882) 84-102.

[3] A. Baker: Transcendental number theory, Cambridge University Press, 1975.

[4] E.W. Barnes: On the theory of the multiple gamma function, Trans. Cambridge Philos. Soc., **19** (1904) 374-425.

[5] H. Cramér: Studien über die Nullstellen der Riemannschen Zetafunktion, Math. Z. **4** (1919) 104-130.

[6] P. Deligne: La conjecture de Weil (I), Inst. Hautes Êtudes Sci. Publ. Math. **43** (1974) 273-307.

[7] A. Deitmar: Schemes over $\mathbf{F}_1$ (preprint, 2004).

[8] A. Deitmar, S. Koyama and N. Kurokawa: Absolute zeta functions, Proc. Japan Acad **84A** (2008) 138-142.

[9] C. Deninger: Local L-factors of motives and regularized determinants, Invent. Math. **107** (1992) 135-150.

[10] T. Fukaya: Hasse zeta functions of non-commutative rings, J. Algebra **208**, (1998) 304-342.

[11] M. Hirano, N. Kurokawa and M. Wakayama: Half zeta functions, J. Ramanujan Math. Soc. **18** (2003) 195-209.

[12] A. Grothendieck: Séminaire de Géométrie Algébrique du Bois-Marie 1965-66 (SGA 5), Lecture Notes in Math. **589** (1977), Springer-Verlag.

[13] S. Kim, S. Koyama and N. Kurokawa: The Riemann hypothesis and functional equations for zeta functions over $\mathbf{F}_1$, Proc. Japan Acad. **85A** (2009) 75-80.

[14] N. Kurokawa: Zeta functions of categories, Proc. Japan Acad. **72** (1996) 221-222.

[15] N. Kurokawa: Multiple zeta functions: an example, Adv. Stud. in Pure Math. **21** (1992) 219-226.

[16] N. Kurokawa and S. Koyama: Normalized double sine functions, Proc. Japan Acad. **79** (2003) 14-18.

[17] N. Kurokawa and S. Koyama: Multiple sine functions, Forum Math. **15** (2003) 839-876.

[18] S. Koyama and N. Kurokawa: Multiple zeta functions: the double sine function and the signed double Poisson summation formula, Compositio Math. **140** (2004) 1176-1190.

[19] S. Koyama and N. Kurokawa: Multiple Euler products, Proc. St. Petersburg Math. Soc. **11** (2005) 123-166 (in Russian). English translation is published in AMS Translations, Series 2 (2006) 101-140.

[20] N. Kurokawa, H. Ochiai and M. Wakayama: Absolute derivations and zeta functions, Documenta Math. 2003 (Kato Extra Volume), 565-584.

[21] N. Kurokawa and M. Wakayama: Absolute tensor products, International Math. Res. Notices, **2004** no. 5. 249-260.

[22] N. Kurokawa and M. Wakayama: Absolute tensor products, International Math. Research Notices **2004**-No.5 (2004), 249-260.

[23] Yu. I. Manin: Lectures on zeta functions and motives (according to Deninger and Kurokawa), Asterisque **228** (1995) 121-163.

[24] M. Schröter: Über Kurokawa-Tensorprodukte von L-Reihen, Schriftenreihe des Mathematischen Instituts der Universität Münster, 3. Serie, Heft 18, April, 1996, 240 pages.

Shin-ya Koyama:

The Institute of Mathematical Sciences, Ewha Womans University,

Daehyun-dong 11-1, Sedaemoon-ku, 120-750, Seoul, South Korea.

Department of Biomedical Engineering, Toyo University,

2100 Kujirai, Kawagoe-shi 350-8585, Japan.

koyama@toyonet.toyo.ac.jp

Nobushige Kurokawa:

Department of Mathematics, Tokyo Institute of Technology,

2-12-1, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan.

kurokawa@math.titech.ac.jp