Research Article

Wataru Takeda* and Shin-ya Koyama

Estimates of Lattice Points in the Discriminant Aspect over Abelian Extension Fields

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Abstract: We estimate the number of relatively r-prime lattice points in K^m with their components having a norm less than x, where K is a number field. The error terms are estimated in terms of x and the discriminant D of the field K, as both x and D grows. The proof uses the bounds of Dedekind zeta functions. We obtain uniform upper bounds as K runs through number fields of any degree under assuming the Lindelöf hypothesis. We also show unconditional results for abelian extensions with a degree less than or equal to 6.

Keywords: Lattice point, Approximation formula, Lindelöf Hypothesis in the discriminant aspect

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1 Introduction

Let K be a number field and $\mathcal{O}_K$ be its ring of integers. We regard an m-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ of $\mathcal{O}_K$ as a lattice point in K^m. We say that a lattice point $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ is *relatively r-prime* for a positive integer r, if there exists no prime ideal $\mathfrak{p}$ such that $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m \subset \mathfrak{p}^r$.

Let $V_m^r(x, K)$ denote the number of relatively r-prime lattice points $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ such that $\mathfrak{N}\mathfrak{a}_\mathfrak{i} \leq x$ $(i = 1, 2, 3, ..., m)$. The behavior of $V_m^r(x, K)$ has long been studied. In 1900, Lehmer [Le00] found that $V_m^1(x, \mathbb{Q}) \sim x^m/\zeta(m)$

***Corresponding author: Wataru Takeda,** Department of Mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan
Shin-ya Koyama, Department of Biomedical Engineering, Toyo University, 2100 Kujirai, Kawagoe, Saitama 350-8585, Japan

as $x \to \infty$. Later in seventies, Benkoski [Be76] generalized it to $V_m^r(x, \mathbb{Q}) \sim x^m/\zeta(rm)$ for any integer $r \geq 1$. The milestone was the work of B. D. Sittinger in 2010 ([St10]). Roughly expressing, he proved in [St10] that

$$V_m^r(x, K) = \frac{c^m}{\zeta_K(rm)} x^m + o(x^m) \qquad (x \to \infty) \tag{1}$$

for any fixed number field K with ζ_K being the Dedekind zeta function of K and with c being a positive constant depending only on K. He actually obtained the error term in (1) in a more concrete form, which we will describe precisely in Theorem 3.1. In his remarkable work, Sittinger dealt with ideals in algebraic integer rings rather than algebraic integers themselves. This idea led us to a new stage for general number fields.

In the previous papers, the first author obtained estimates of the error term

$$E_m^r(x, K) = V_m^r(x, K) - \frac{c^m}{\zeta_K(rm)} x^m$$

as follows:

Theorem ([Ta17][Ta17b]). *Let $n = [K : \mathbf{Q}]$. It holds that*

$$E_m^r(x, K) = \begin{cases} O\left(x^{m-\alpha(n)}(\log x)^{\beta(n)}\right) & \text{if } rm \geq 3, \\ O\left(x^{2-\alpha(n)}(\log x)^{2\beta(n)+1}\right) & \text{if } (r, m) = (1, 2), \\ O\left(x^{1-\frac{\alpha(n)}{2}}(\log x)^{2\beta(n)}\right) & \text{if } (r, m) = (2, 1) \end{cases} \qquad (x \to \infty) \tag{2}$$

with

$$\alpha(n) = \begin{cases} \frac{2}{n} - \frac{8}{n(5n+2)} & (3 \leq n \leq 6), \\ \frac{2}{n} - \frac{3}{2n^2} & (7 \leq n \leq 9), \\ \frac{3}{n+6} - \varepsilon & (n \geq 10) \end{cases} \qquad \text{and} \qquad \beta(n) = \begin{cases} \frac{10}{5n+2} & (3 \leq n \leq 6), \\ \frac{2}{n} & (7 \leq n \leq 9), \\ 0 & (n \geq 10). \end{cases}$$

Moreover if we assume the Lindelöf hypothesis for $\zeta_K(s)$, it holds for all $\varepsilon > 0$ that

$$E_m^r(x, K) = \begin{cases} O\left(x^{\frac{3}{4}+\varepsilon}\right) & \text{if } (r, m) = (2, 1), \\ O\left(x^{m-\frac{1}{2}+\varepsilon}\right) & \text{otherwise}, \end{cases}$$

as $x \to \infty$.

The estimates (2) improves the original bound of Sittinger [St10] for any number field K with $[K : \mathbf{Q}] \geq 3$.

Our goal is to study another aspect of $E_m^r(x, K)$. Our chief concern is the behavior of $E_m^r(x, K)$ with the field K being varied. We express the bounds in terms of the absolute value $D = D_K$ of the discriminant of K as well as x, and our results are described as both x and D go to infinity.

Our proof heavily uses estimates of Dedekind zeta functions. In particular, the Lindelöf hypotheses in the discriminant aspect gives the conjectural best estimate. We also show unconditional results by using known bounds of L-functions.

After the preliminaries in Sections 2 and 3, we first present a conditional result under the Lindelöf hypothesis for $\zeta_K(s)$ in the aspect of both $\Im(s)$ and D in Section 4. Theorem 4.1 below asserts under the Lindelöf hypothesis that

$$\begin{aligned} V_m^r(x,K) &= \frac{c^m}{\zeta_K(rm)} x^m \\ &\quad + \begin{cases} O\left(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{2\varepsilon - \frac{m-1}{2}}\right) & \textit{if } rm=2, \text{ or } r=3, m=1 \text{ and } \varepsilon < \frac{1}{10}, \\ O\left(x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon-\frac{m-1}{2}}\right) & \text{otherwise,} \end{cases} \end{aligned}$$

as x, $D \to \infty$, where the field K runs through all number fields with $x^{1-2\varepsilon} > D^{1+2\varepsilon}$.

Next we show unconditional results by restricting the degree of K to be less than or equal to 6. We prove in Theorem 4.2 that

$$E_m^r(x,K) = \begin{cases} O\left(x^{\frac{1}{r}\left(\frac{23929}{15960}+\frac{89n}{1140}+\varepsilon\right)} D^{\frac{31}{95}-\frac{m-1}{2}}\right) & \text{if } rm=2, \\ O\left(x^{m+\frac{89n}{1140}-\frac{7991}{15960}+\varepsilon} D^{\frac{31}{190}-\frac{m-1}{2}}\right) & \text{otherwise,} \end{cases}$$

as x, $D \to \infty$, where K runs through abelian extensions with $x^{\frac{1}{753}+\varepsilon} > D$ and $[K:\mathbb{Q}] \le 6$.

Removing the assumption that K is abelian is an open question. Some progress toward this problem will be studied in the forthcoming paper [Ta17c].

2 The Lindelöf hypothesis in the discriminant aspect

In this section, we introduce the Lindelöf hypothesis in the discriminant aspect which is implied by that in the aspect of the analytic conductor.

For an L-function $L(s,\chi)$ having an Euler product of degree d, the analytic conductor $\mathfrak{q}(s,\chi)$ is defined as

$$\mathfrak{q}(s,\chi) = q(\chi) \prod_{j=1}^{d} (|s+\kappa_j|+3), \tag{3}$$

where $q(\chi)$ is the conductor of χ and κ_j is the local parameters of $L(s,\chi)$ at infinity. For the details for the definition of the analytic conductor, one can see Iwaniec and Kowalski's book [IK04].

The Dedekind zeta function ζ_K over K is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}\mathfrak{a}^s}$$

with the sum taken over all nonzero ideals of $\mathcal{O}_K$. It is known that ζ_K has an Euler product of degree $n = [K : \mathbf{Q}]$.

The Lindelöf hypothesis is extended for many L-functions. In Iwaniec and Kowalski's book [IK04], this hypothesis is written as

The Lindelöf Hypothesis in the analytic conductor aspect.

Let $L(s,\chi)$ be an L-function, then for every $\varepsilon > 0$,

$$L\left(\frac{1}{2} + it, \chi\right) = O\left(\mathfrak{q}(s,\chi)^{\varepsilon}\right) \qquad (\mathfrak{q} \to \infty),$$

where $\mathfrak{q}(s,\chi)$ is the analytic conductor and the constant implied in O depends on ε alone.

Put $D = |d_K|$, where d_K is the discriminant of K. The analytic conductor of $\zeta_K(s) = L(s,\chi_K)$ is given by

$$\mathfrak{q}(s,\chi_K) = D(|t|+3)^{r_1+r_2}(|t+1|+3)^{r_2}, \tag{4}$$

where r_1 is the number of real embeddings of K and r_2 is the number of pairs of complex embeddings. It is well known that $n = r_1 + 2r_2$. The Lindelöf hypothesis in the discriminant aspect is expressed as follows.

The Lindelöf Hypothesis in the discriminant aspect.

Let D be the absolute value of the discriminant of K, and assume $|t| > 1$. Then for every $\varepsilon > 0$,

$$\zeta_K\left(\frac{1}{2} + it\right) = O\left((D|t|^n)^{\varepsilon}\right) \qquad (D|t|^n \to \infty), \tag{5}$$

where the constant implied in O depends on ε alone.

The conjecture (5) implies behaviors of $\zeta_K\left(\frac{1}{2} + it\right)$ with one of t and D being fixed and with the other variable growing.

Partial results towards the conjecture (5) are known in the sense that some nontrivial exponents are obtained with both t and D growing under a certain condition. Namely, Huxley and Watt showed that for primitive Dirichlet characters χ modulo p

$$L\left(\frac{1}{2}+it,\chi\right)=O\left(|t|^{\frac{89}{570}+\varepsilon}p^{\frac{31}{190}}\right)\qquad (|t|,p\to\infty) \tag{6}$$

with $p<|t|^{\frac{2}{753}}$ for all $\varepsilon>0$ [HW00].

When K is an abelian extension field, we have a factorization of the Dedekind zeta function as $\zeta_K(s)=\zeta_{\mathbf{Q}}(s)\prod_\chi L(s,\chi)$. Here χ runs through Dirichlet characters so that the product of their conductors is equal to D. Bourgain showed that

$$\zeta_{\mathbf{Q}}\left(\frac{1}{2}+it\right)=O\left(|t|^{\frac{13}{84}+\varepsilon}\right), \tag{7}$$

as $|t|\to\infty$ for all $\varepsilon>0$ [Bo17]. Since the Dedekind zeta function ζ_K has an Euler product of degree $n=[K:\mathbf{Q}]$, it holds from (6) and (7) that for any $\varepsilon>0$

$$\zeta_K\left(\frac{1}{2}+it\right)=O\left(|t|^{\frac{89n}{570}-\frac{11}{7980}+\varepsilon}D^{\frac{31}{190}}\right)\qquad (|t|,D\to\infty), \tag{8}$$

as K runs through all abelian extension fields with $D<|t|^{\frac{2}{753}}$.

3 Auxiliary Theorems

In this section we prepare auxiliary theorems which are necessary for showing the main theorem. Let $I_K(x)$ be the number of ideals of $\mathcal{O}_K$ with their ideal norm less than or equal to x.

Put $n=[K:\mathbf{Q}]$. In [St10], Sittinger used the estimate $I_K(x)=cx+O\left(x^{1-\frac{1}{n}}\right)$ $(x\to\infty)$ to show the following theorem:

Theorem 3.1 (Sittinger [St10])**.** *It holds that*

$$V_m^r(x,K)=\frac{c^m}{\zeta_K(rm)}x^m+\begin{cases} O(x^{m-\frac{1}{n}}) & \text{if } m\geq 3,\ \text{or } m=2 \text{ and } r\geq 2,\\ O(x^{2-\frac{1}{n}}\log x) & \text{if } m=2 \text{ and } r=1,\\ O(x^{1-\frac{1}{n}}\log x) & \text{if } m=1 \text{ and } \frac{n(r-2)}{r-1}=1,\\ O(x^{1-\frac{1}{n}}) & \text{if } m=1 \text{ and } \frac{n(r-2)}{r-1}>1,\\ O(x^{\frac{1}{r}(2-\frac{1}{n})}) & \text{if } m=1 \text{ and } \frac{n(r-2)}{r-1}<1 \end{cases}$$

as $x\to\infty$.

In the previous papers, the first author also used better estimates of $I_K(x)$ and improved estimates of $E_m^r(x,K)$ as $x\to\infty$.

Here we are estimating $I_K(x)$ as $x,\ D\to\infty$ under the hypothesis (5).

Theorem 3.2. *Assume the hypothesis* (5)*. Then for every* $0 < \varepsilon < \frac{1}{2}$ *we have*

$$I_K(x) = cx + O\left(x^{\frac{1}{2}+\varepsilon} D^{\varepsilon}\right) \qquad (x, D \to \infty),$$

where K *runs through all number fields with* $x^{1-2\varepsilon} > D^{1+2\varepsilon}$*. Here the constant* c *is defined by*

$$c = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} \tag{9}$$

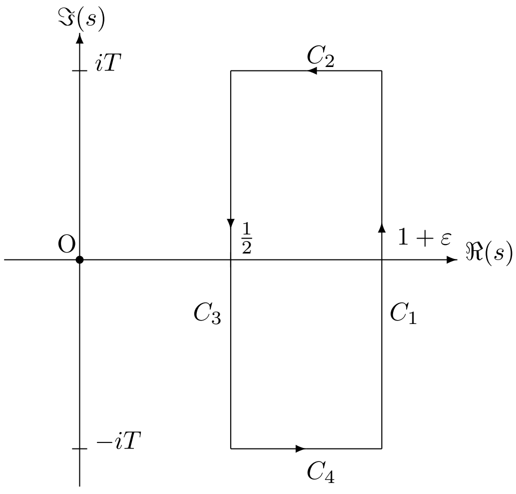
with h*,* R*, and* w *being the class number, the regulator, and the number of roots of unity in the integer ring of* K*, respectively.*

Proof. It suffices to show that $I_K(x) = cx + O\left(x^{\frac{1}{2}+\varepsilon} D^{\varepsilon}\right)$ for any half-integer $x = n+\frac{1}{2}$ with n a positive integer, since it holds for any real number $y \in [n, n+1)$ that $I_K(x) = I_K(y)$.

We consider the integral

$$\frac{1}{2\pi i}\int_C \zeta_K(s)\frac{x^s}{s}\, ds,$$

where C is the contour $C_1 \cup C_2 \cup C_3 \cup C_4$ in the following figure.



In a way similar to the well-known proof of Perron's formula, we estimate

$$\frac{1}{2\pi i}\int_{C_1} \zeta_K(s)\frac{x^s}{s}\, ds = I_K(x) + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

We can select the large T, so that the O-term in the right hand side is sufficiently small. For estimating the left hand side by using the hypotheses (5), we divide it into the integrals over C_2, C_3 and C_4. First we calculate the integral over C_3 as

$$\left|\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\right| = \left|\frac{1}{2\pi i}\int_{T}^{-T}\zeta_K\left(\frac{1}{2}+it\right)\frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it}i\,dt\right|$$
$$\leq \frac{1}{2\pi}\int_{-T}^{T}\left|\zeta_K\left(\frac{1}{2}+it\right)\right|\frac{x^{\frac{1}{2}}}{\left|\frac{1}{2}+it\right|}\,dt.$$

By the hypothesis (5) we conclude that

$$\left|\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\right| = O\left(\int_{-T}^{T}(T^nD)^{\varepsilon}\frac{x^{\frac{1}{2}}}{\left|\frac{1}{2}+it\right|}\,dt\right)$$
$$= O\left(\int_{1}^{T}(T^nD)^{\varepsilon}\frac{x^{\frac{1}{2}}}{t}\,dt + \int_{-1}^{1}(T^nD)^{\varepsilon}\frac{x^{\frac{1}{2}}}{\frac{1}{2}+it}\,dt\right)$$
$$= O\left(x^{\frac{1}{2}}(T^nD)^{\varepsilon}\right).$$

Next we consider the integrals over C_2 and C_4. It holds by (5) that their sum is estimated as

$$\left|\frac{1}{2\pi i}\int_{C_2\cup C_4}\zeta_K(s)\frac{x^s}{s}\,ds\right| \leq \frac{1}{2\pi}\int_{\frac{1}{2}}^{1+\varepsilon}\left|\zeta_K(\sigma+iT)\right|\frac{x^{\sigma}}{T}\,d\sigma + \frac{1}{2\pi}\int_{\frac{1}{2}}^{1+\varepsilon}\left|\zeta_K(\sigma-iT)\right|\frac{x^{\sigma}}{T}\,d\sigma$$
$$= O\left(\int_{\frac{1}{2}}^{1+\varepsilon}(T^nD)^{\varepsilon}\frac{x^{\sigma}}{T}\,d\sigma\right)$$
$$= O\left(\frac{x^{1+\varepsilon}D^{\varepsilon}}{T^{1-n\varepsilon}}\right).$$

By Cauchy's residue theorem we get

$$\frac{1}{2\pi i}\int_{C}\zeta_K(s)\frac{x^s}{s}\,ds = \rho x,$$

where ρ is the residue of $\zeta_K(s)$ at $s=1$. But it is known that $\rho = c$ ([La94]).

By gathering all results above, we reach

$$I_K(x) = cx + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{1}{2}}T^{n\varepsilon}D^{\varepsilon}\right) + O\left(\frac{x^{1+\varepsilon}D^{\varepsilon}}{T^{1-n\varepsilon}}\right).$$

When we select $T = x^{\frac{1}{2}+\varepsilon}$, this becomes

$$I_K(x) = cx + O\left(x^{\frac{1}{2}+\varepsilon}D^{\varepsilon}\right).$$

This proves the theorem. □

Next we obtain unconditional estimates of $I_K(x)$ by using (8). The following theorem is shown in a way similar to the proof of Theorem 3.2 with $T^{n\varepsilon}D^{\varepsilon}$ replaced by $T^{\frac{89n}{570}-\frac{11}{7980}+\varepsilon}D^{\frac{31}{190}}$.

Theorem 3.3. *Let S be a subset of $\{K : \text{abelian extension field} \mid [K:\mathbf{Q}] \le 6\}$ and $n = \max\{[K:\mathbf{Q}] \;:\; K \in S\}$. Then for every $\varepsilon > 0$, we have*

$$I_K(x) = cx + O\left(x^{\frac{89n}{1140}+\frac{7969}{15960}+\varepsilon}D^{\frac{31}{190}}\right) \qquad (x, D \to \infty).$$

where K runs through elements in S satisfying that $x^{\frac{1}{753}+\varepsilon} > D$.

4 Main results

We estimated $I_K(x)$ in the last section. Theorems 3.2 and 3.3 will play a crucial role in our computing the number of relatively r-prime lattice points by the relation

$$V_m^r(x, K) = \sum_{\mathfrak{N}\mathfrak{a} \le x^{\frac{1}{r}}} \mu(\mathfrak{a}) I_K\left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^m, \tag{10}$$

where $\mu(\mathfrak{a})$ is the Möbius function defined as

$$\mu(\mathfrak{a}) \stackrel{\text{def}}{=} \begin{cases} 0 & \textit{if } \mathfrak{a} \subset \mathfrak{p}^2 \text{ for some prime ideal } \mathfrak{p}, \\ 1 & \textit{if } \mathfrak{a} = 1, \\ (-1)^s & \textit{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s, \text{ where } \mathfrak{p}_1, \ldots, \mathfrak{p}_s \text{ are distinct prime ideals.} \end{cases}$$

Theorem 4.1. *If we assume the hypothesis* (5)*, then it holds for all $0 < \varepsilon < \frac{1}{2}$ that*

$$V_m^r(x, K) = \frac{c^m}{\zeta_K(rm)} x^m + \begin{cases} O\left(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)}D^{2\varepsilon - \frac{m-1}{2}}\right) & \textit{if } rm = 2, \;\textit{ or } r = 3, m = 1 \textit{ and } \varepsilon < \frac{1}{10}, \\ O\left(x^{m-\frac{1}{2}+\varepsilon}D^{\varepsilon - \frac{m-1}{2}}\right) & \textit{otherwise,} \end{cases}$$

as $x, D \to \infty$, where K runs through all number fields with $x^{1-2\varepsilon} > D^{1+2\varepsilon}$.

Proof. We use the identity (10) to compute as follows. Theorem 3.2 and the binomial theorem lead to

$$V_m^r(x,K) = \sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \mu(\mathfrak{a}) \left(\frac{cx}{\mathfrak{Na}^r} + O\left(\left(\frac{x}{\mathfrak{Na}^r} \right)^{\frac{1}{2}+\varepsilon} D^{\varepsilon} \right) \right)^m$$

$$= (cx)^m \sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} + O\left(\sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \left(\frac{cx}{\mathfrak{Na}^r} \right)^{m-1} \left(\frac{x}{\mathfrak{Na}^r} \right)^{\frac{1}{2}+\varepsilon} D^{\varepsilon} \right).$$

From the definition of c (9)

$$V_m^r(x,K) = (cx)^m \sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} + O\left(\sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \left(\frac{D^{-\frac{1}{2}}x}{\mathfrak{Na}^r} \right)^{m-1} \left(\frac{x}{\mathfrak{Na}^r} \right)^{\frac{1}{2}+\varepsilon} D^{\varepsilon} \right)$$

$$= (cx)^m \sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} + O\left(\sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \left(\frac{x}{\mathfrak{Na}^r} \right)^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon-\frac{m-1}{2}} \right).$$

By using the fact that

$$\sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} = \frac{1}{\zeta_K(rm)},$$

we get

$$V_m^r(x,K) = \frac{c^m}{\zeta_K(rm)} - (cx)^m \sum_{\mathfrak{Na} > x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} + O\left(\sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \left(\frac{x}{\mathfrak{Na}^r} \right)^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon-\frac{m-1}{2}} \right).$$

The first term $(cx)^m/\zeta_K(rm)$ agrees to the principal term of $V_m^r(x,K)$ ([St10]). Thus $E_m^r(x,K)$ is expressed as

$$E_m^r(x,K) = -(cx)^m \sum_{\mathfrak{Na} > x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{Na}^{rm}} + O\left(\sum_{\mathfrak{Na} \le x^{\frac{1}{r}}} \left(\frac{x}{\mathfrak{Na}^r} \right)^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon-\frac{m-1}{2}} \right).$$

Now we estimate the behavior of the first sum in the right hand side as $x, D \to \infty$. From Theorem 3.2, it follows that $I_K(x) - I_K(x-1) = O(x^{\frac{1}{2}+\varepsilon} D^{\varepsilon})$, and so we have

$$(cx)^m \sum_{\mathfrak{N}\mathfrak{a} > x^{\frac{1}{r}}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} = O\left(\left(D^{-\frac{1}{2}} x\right)^m \int_{x^{\frac{1}{r}}}^{\infty} \frac{y^{\frac{1}{2}+\varepsilon} D^{\varepsilon}}{y^{rm}}\, dy \right)$$
$$= O\left(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{\varepsilon - \frac{m}{2}} \right).$$

Next we deal with the second sum. Again using $I_K(x) - I_K(x-1) = O(x^{\frac{1}{2}+\varepsilon} D^{\varepsilon})$, we have

$$\sum_{\mathfrak{N}\mathfrak{a} \le x^{\frac{1}{r}}} \left(\frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}}$$
$$= O\left(x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}} \left(1 + \int_1^{x^{\frac{1}{r}}} \frac{y^{\frac{1}{2}+\varepsilon} D^{\varepsilon}}{y^{r(m-\frac{1}{2}+\varepsilon)}}\, dy \right) \right)$$
$$= \begin{cases} O\left(x^{\frac{3}{2}+\varepsilon} D^{\varepsilon - \frac{1}{2}} + x^{\frac{3}{2}+\varepsilon} \log x D^{2\varepsilon - \frac{1}{2}} \right) & \textit{if } r = 1, m = 2, \\ O\left(x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}} + x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{2\varepsilon - \frac{m-1}{2}} \right) & \text{otherwise.} \end{cases}$$

Comparing $x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}}$ and $x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{2\varepsilon - \frac{m-1}{2}}$ to find out which is greater, we obtain

$$\sum_{\mathfrak{N}\mathfrak{a} \le x^{\frac{1}{r}}} \left(\frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}}$$
$$= \begin{cases} O\left(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{2\varepsilon - \frac{m-1}{2}} \right) & \textit{if } rm = 2, \text{ or } r = 3, m = 1 \text{ and } \varepsilon < \frac{1}{10}, \\ O\left(x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}} \right) & \text{otherwise.} \end{cases}$$

Hence we get

$$V_m^r(x, K) = \frac{c^m}{\zeta_K(rm)} x^m$$
$$+ \begin{cases} O\left(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)} D^{2\varepsilon - \frac{m-1}{2}} \right) & \textit{if } rm = 2, \text{ or } r = 3, m = 1 \text{ and } \varepsilon < \frac{1}{10}, \\ O\left(x^{m-\frac{1}{2}+\varepsilon} D^{\varepsilon - \frac{m-1}{2}} \right) & \text{otherwise.} \end{cases}$$

This proves the theorem. □

Theorem 3.3 gives our conclusion on the estimate of $E_m^r(x, K)$, whose proof is similar to that of Theorem 4.1.

Theorem 4.2. *Let S be a subset of $\{K : \text{abelian extension field} \mid [K : \mathbf{Q}] \le 6\}$ and $n = \max\{[K : \mathbf{Q}] \ : \ K \in S\}$. Then for every $\varepsilon > 0$,*

$$V_m^r(x, K) = \frac{c^m}{\zeta_K(rm)} x^m + \begin{cases} O\left(x^{\frac{1}{r}\left(\frac{23929}{15960} + \frac{89n}{1140} + \varepsilon\right)} D^{\frac{31}{95} - \frac{m-1}{2}}\right) & \text{if } rm = 2, \\ O\left(x^{m + \frac{89n}{1140} - \frac{7991}{15960} + \varepsilon} D^{\frac{31}{190} - \frac{m-1}{2}}\right) & \text{otherwise,} \end{cases}$$

as $x, D \to \infty$, where K runs through elements in S satisfying that $x^{\frac{1}{753}+\varepsilon} > D$.

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