Euler Product Expression of Triple Zeta Functions

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Abstract

We construct multiple zeta functions considered as absolute tensor products of usual zeta functions. We establish Euler product expressions for triple zeta functions $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)$ with p,q,r distinct primes, via multiple sine functions by using the signatured Poisson summation formula.

1. Introduction and Main Theorem

Let

$$Z_j(s)=\prod_{\rho\in\mathbb{C}}(s-\rho)^{m_j(\rho)}$$

be "zeta functions" expressed as regularized products in the notation of Deninger [D1], [D2], where $m_j:\mathbb{C}\to\mathbb{Z}$ denotes the multiplicity function for $j=1,\ldots,r$. (Later we will specify "zeta functions" to be treated.) Then, as in the paper [K] we define the absolute tensor product $Z_1(s)\otimes\cdots\otimes Z_r(s)$ as

$$Z_1(s)\otimes\cdots\otimes Z_r(s)=\coprod_{\rho_1,\ldots,\rho_r\in\mathbb{C}}(s-(\rho_1+\cdots+\rho_r))^{m(\rho_1,\ldots,\rho_r)},$$

where

$$m(\rho_1,\ldots,\rho_r):=m_1(\rho_1)\cdots m_r(\rho_r)\times\begin{cases}1 & \text{if } \mathrm{Im}(\rho_1),\ldots,\mathrm{Im}(\rho_r)\geqslant 0,\\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_1),\ldots,\mathrm{Im}(\rho_r)<0,\\ 0 & \text{otherwise.}\end{cases}$$

We define Hasse zeta functions $\zeta(s,A)$ for commutative rings A as

$$\zeta(s,A):=\prod_{\mathfrak{m}}(1-N(\mathfrak{m})^{-s})^{-1},$$

where $\mathfrak{m}$ runs over maximal ideals of A, and $N(\mathfrak{m}):=\#(A/\mathfrak{m})$.

The following result about the absolute tensor product of Hasse zeta functions is known:

Theorem 1.1. *([KoKu] Theorem 1.1) The following expressions hold in* $\mathrm{Re}(s)>0$ *with some polynomial* $Q(s)$ *of degree at most two:*
(1) When $p\neq q$*, it holds that*

$$\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)=e^{Q_{p,q}(s)}(1-p^{-s})^{\frac{1}{2}}(1-q^{-s})^{\frac{1}{2}}\times$$
$$\exp\left(\frac{1}{2i}\sum_{k=1}^{\infty}\frac{\cot\left(\pi k\frac{\log p}{\log q}\right)}{k}p^{-ks}+\frac{1}{2i}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log q}{\log p}\right)}{n}q^{-ns}\right).$$

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(2) When $p=q$, it holds that

$$\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_p)=e^{Q_{p,p}(s)}(1-p^{-s})^{1-\frac{is\log p}{2\pi}}\exp\left(-\frac{\mathrm{Li}_2(p^{-s})}{2\pi i}\right),$$

where

$$\mathrm{Li}_k(x):=\sum_{n=1}^{\infty}\frac{x^n}{n^k}.$$

In this paper we treat $\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)$ when p,q,r are distinct primes. We prove the following theorem, which is similar to Theorem 1.1(1):

Theorem 1.2. *Suppose that p,q,r are distinct primes and $s\in\mathbb{C}$ satisfies $\mathrm{Re}(s)>0$. Then,*

$$\begin{aligned}
&\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\\
&=e^{Q_{p,q,r}(s)}(1-p^{-s})^{-\frac{1}{4}}(1-q^{-s})^{-\frac{1}{4}}(1-r^{-s})^{-\frac{1}{4}}\\
&\quad\exp\left(-\frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi n_1\frac{\log p}{\log q}\right)\cot\left(\pi n_1\frac{\log p}{\log r}\right)p^{-n_1s}\right.\\
&\qquad-\frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi n_2\frac{\log q}{\log p}\right)\cot\left(\pi n_2\frac{\log q}{\log r}\right)q^{-n_2s}\\
&\qquad-\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi n_3\frac{\log r}{\log p}\right)\cot\left(\pi n_3\frac{\log r}{\log q}\right)r^{-n_3s}\\
&\qquad+\frac{i}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(\cot\left(\pi n_1\frac{\log p}{\log q}\right)+\cot\left(\pi n_1\frac{\log p}{\log r}\right)\right)p^{-n_1s}\\
&\qquad+\frac{i}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi n_2\frac{\log q}{\log p}\right)+\cot\left(\pi n_2\frac{\log q}{\log r}\right)\right)q^{-n_2s}\\
&\qquad\left.+\frac{i}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi n_3\frac{\log r}{\log p}\right)+\cot\left(\pi n_3\frac{\log r}{\log q}\right)\right)r^{-n_3s}\right),
\end{aligned}$$

where $Q_{p,q,r}(s)$ is a polynomial of degree at most three, which depends on p,q,r.

2. Triple Poisson Summation Formula with Signature

Definition 2.1. $\alpha\in\mathbb{R}$ is said to be *generic, if*

$$\lim_{m\to\infty}\|m\alpha\|^{\frac{1}{m}}=1,$$

where we put $\|x\|:=\min\{|x-n|:n\in\mathbb{Z}\}$ for $x\in\mathbb{R}$.

Lemma 2.2. *If $\alpha\in\mathbb{Q}$, then α is not generic.*

Proof. Let $\alpha=a/b$ with $a\in\mathbb{Z}$ and $b\in\mathbb{Z}_{>0}$. Then $\|(bm)\alpha\|=0$ for all $m\in\mathbb{Z}$. Hence α is not generic. $\square$

Lemma 2.3. *(Baker, [B]Theorem 3.1) Let α and β be in $\overline{\mathbb{Q}}\cap\mathbb{R}_{>0}$. If $\frac{\log\alpha}{\log\beta}\notin\mathbb{Q}$, then $\frac{\log\alpha}{\log\beta}$ is generic.*

Lemma 2.4. *Let α and β be generic. Then*

$$\sum_{n=1}^{\infty}\cot(\pi n\alpha)\cot(\pi n\beta)x^n \tag{2.1}$$

converges absolutely in $|x| < 1$.

Proof. Since $\alpha,\ \beta$ are generic, $\|n\alpha\|^{-1},\ \|n\beta\|^{-1} = O(e^{\varepsilon n})$ as $n \to \infty$ for any $\varepsilon > 0$. Since $\cot(\pi x) \sim 1/(\pi x)$ $(x \to 0)$, $\cot(\pi n\alpha)\cot(\pi n\beta) = O(e^{2\varepsilon n})$ $(n \to \infty)$. Hence the radius of convergence of (2.1), say R, satisfies

$$R = \limsup_{n\to\infty} \left|\frac{1}{\cot(\pi n\alpha)\cot(\pi n\beta)}\right|^{\frac{1}{n}} \geqslant e^{-2\varepsilon}.$$

Since $\varepsilon > 0$ is arbitary, $R \geqslant 1$. □

LEMMA 2.5. *Let n, m be positive integers and a, b be positive real numbers such that a/b is generic. Then, for any $\varepsilon > 0$ we have*

$$|na - mb|^{-1} \ll_{a,b,\varepsilon} m^{-1} e^{\varepsilon n}$$

Proof. For all $n, m \in \mathbb{Z}_{>0}$ we have

$$|na - mb| = mb\left|\frac{na}{mb} - 1\right| \geqslant mb\left(\left[\frac{na}{b}\right] + 1\right)^{-1}\left\|\frac{na}{b}\right\|,$$

where $[x] := \max\{k \in \mathbb{Z} : k \leqslant x\}$. Since a/b is generic, for any $\varepsilon > 0$ we have

$$\left\|n\frac{a}{b}\right\|^{-1} \ll_{a,b,\varepsilon} e^{\varepsilon n}.$$

These show the lemma. □

For $f \in L^1(\mathbb{R})$, we denote by $\widetilde{f}$ the Fourier transform of f:

$$\widetilde{f}(x) := \int_{-\infty}^{\infty} f(t) e^{itx} dt.$$

Let $H(t)$ be an even regular function on $\{z = x + iy : x \in \mathbb{R}, y \in (-R, R)\}$, which satisfies (i) and (ii):
(i)There exists $\delta > 0$ such that $H(t) = O(t^{-3-\delta})$ $(|t| \to \infty)$
(ii)There exists $\mu \in (0,1)$ such that $\widetilde{H}(x) = O(\mu^x)$ $(x \to \infty)$

$h(t)$ and $H_\alpha(t)$ denote $h(t) := H(t/i)$, $H_\alpha(t) := h(\alpha + it)$, respectively.

LEMMA 2.6. *Let $\alpha \in (-R, R)$ and $x \in \mathbb{R}$. Then,*
(1)$\widetilde{H_\alpha}(x) = e^{-\alpha x}\widetilde{H}(x)$.
(2)$\widetilde{tH_\alpha(t)}(0) = i\alpha\widetilde{H}(0)$.

Proof. (1)Applying Cauchy's theorem to $H(t)e^{itx}$, we have

$$\int_{C_T} H(t)e^{itx} dt = 0,$$

where

$$C_T := \partial\{z \in \mathbb{C} | -T < \mathrm{Re}(z) < T,\ \min\{-\alpha, 0\} < \mathrm{Im}(z) < \max\{-\alpha, 0\}\}.$$

Considering the limit $T \to \infty$, (1) follows.

(2)Considering the same as (1), we have

$$\int_{-\infty}^{\infty} tH(t)dt = \int_{-\infty}^{\infty} (t - i\alpha)H_\alpha(t)dt. \tag{2.2}$$

Since $tH(t)$ is an odd function, left side of (2.2) equals to 0. So

$$\widetilde{tH_\alpha(t)}(0) = i\alpha\widetilde{H_\alpha}(0)$$

$$= i\alpha\widetilde{H}(0).$$

$\square$

Theorem 2.7. *Let $a, b, c \in \mathbb{R}$ be positive real numbers such that $a/b, b/c, c/a$ and their inverses are generic. Then we have*

$$\begin{aligned}
&\sum_{n_1,n_2,n_3>0} H\left(2\pi\left(\frac{n_1}{a}+\frac{n_2}{b}+\frac{n_3}{c}\right)\right)+\frac{1}{2}\left(\sum_{n_1,n_2>0} H\left(2\pi\left(\frac{n_1}{a}+\frac{n_2}{b}\right)\right)\right.\\
&\quad\left.+\sum_{n_2,n_3>0} H\left(2\pi\left(\frac{n_2}{b}+\frac{n_3}{c}\right)\right)+\sum_{n_1,n_3>0} H\left(2\pi\left(\frac{n_1}{a}+\frac{n_3}{c}\right)\right)\right)\\
&\quad+\frac{1}{4}\left(\sum_{n_1>0} H\left(2\pi\frac{n_1}{a}\right)+\sum_{n_2>0} H\left(2\pi\frac{n_2}{b}\right)+\sum_{n_3>0} H\left(2\pi\frac{n_3}{c}\right)\right)+\frac{1}{8}H(0)\\
&=-\frac{a}{8\pi}\sum_{n_1>0}\cot\left(\pi\frac{n_1 a}{b}\right)\cot\left(\pi\frac{n_1 a}{c}\right)\widetilde{H}(n_1 a)\\
&\quad-\frac{b}{8\pi}\sum_{n_2>0}\cot\left(\pi\frac{n_2 b}{c}\right)\cot\left(\pi\frac{n_2 b}{a}\right)\widetilde{H}(n_2 b)\\
&\quad-\frac{c}{8\pi}\sum_{n_3>0}\cot\left(\pi\frac{n_3 c}{a}\right)\cot\left(\pi\frac{n_3 c}{b}\right)\widetilde{H}(n_3 c)\\
&\quad+\frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0)-\frac{abc}{32\pi^3}\widetilde{H}''(0). \qquad (2.3)
\end{aligned}$$

Proof. Put $Z_k(s)=\sinh\left(\dfrac{ks}{2}\right)$ $(k=a,b,c)$. Let D_T be the region defined by

$$D_T := \{s\in\mathbb{C} : |s|>\alpha,\ |\operatorname{Re}(s)|<\alpha,\ 0<\operatorname{Im}(s)<T\},$$

where $0<\alpha<\min\left\{\dfrac{2\pi}{a},\ \dfrac{2\pi}{b},\ \dfrac{2\pi}{c}\right\}$.

By Cauchy's theorem we have

$$\begin{aligned}
&\sum_{0<\operatorname{Im}(\rho_1),\operatorname{Im}(\rho_2),\operatorname{Im}(\rho_3)<T} h(\rho_a+\rho_b+\rho_c)\\
&=\frac{1}{(2\pi i)^3}\int_{\partial D_T}\int_{\partial D_T}\int_{\partial D_T} h(s_1+s_2+s_3)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\frac{Z_c'}{Z_c}(s_3)ds_1ds_2ds_3, \qquad (2.4)
\end{aligned}$$

where ρ_k denotes the zeros of $Z_k(s)(k=a,b,c)$, and the contour ∂D_T is taken counterclockwise. Considering $T\to\infty$ in (2.4), we have

$$\begin{aligned}
&\sum_{0<\operatorname{Im}(\rho_1),\operatorname{Im}(\rho_2),\operatorname{Im}(\rho_3)} h(\rho_1+\rho_2+\rho_3)\\
&=\frac{1}{(2\pi i)^3}\int_{\partial D}\int_{\partial D}\int_{\partial D} h(s_1+s_2+s_3)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\frac{Z_c'}{Z_c}(s_3)ds_1ds_2ds_3, \qquad (2.5)
\end{aligned}$$

where

$$D := \{s\in\mathbb{C} : |\operatorname{Re}(s)|<\alpha,\ |s|>\alpha,\ \operatorname{Im}(s)>0\}.$$

We decompose $\partial D = C_1\cup C_2\cup C_3$ with

$$C_1 := \{s\in\partial D : \operatorname{Re}(s)=-\alpha\},$$
$$C_2 := \{s\in\partial D : |s|=\alpha\},$$

$$C_3 := \{s \in \partial D : \mathrm{Re}(s) = \alpha\}.$$

We compute each triple integral $I_{i_1 i_2 i_3} = \int_{C_{i_1}}\int_{C_{i_2}}\int_{C_{i_3}}$ in (2.5).

First we calculate $I_{i_1 i_2 i_3}$ with $(i_1, i_2, i_3) \in \{1, 3\}^3$.

$$I_{333} = \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\infty\int_0^\infty h(3\alpha + i(t_1 + t_2 + t_3)) \\ \frac{Z_a'}{Z_a}(\alpha + it_1)\frac{Z_b'}{Z_b}(\alpha + it_2)\frac{Z_c'}{Z_c}(\alpha + it_3)dt_1dt_2dt_3. \tag{2.6}$$

Since

$$\frac{Z_k'}{Z_k}(s) = \frac{k}{2} + k\sum_{n=1}^{\infty} e^{-kns} \tag{2.7}$$

for $k = a, b, c$ and $\mathrm{Re}(s) > 0$, (2.6) turns to

$$I_{333} = \sum_{n_1,n_2,n_3 \geqslant 0} \varepsilon_{n_1,n_2,n_3}\int_0^\infty\int_0^\infty\int_0^\infty H_{3\alpha}(t_1 + t_2 + t_3) \\ e^{-n_1a(\alpha+it_1)}e^{-n_2b(\alpha+it_2)}e^{-n_3c(\alpha+it_3)}dt_1dt_2dt_3,$$

where

$$\varepsilon_{n_1,n_2,n_3} := \left(\frac{1}{2}\right)^{\sharp\{i\in\{1,2,3\}:n_i=0\}}.$$

We replace t_3 with $t = t_1 + t_2 + t_3$ to get

$$I_{333} = \frac{1}{8\pi^3}\sum_{n_1,n_2,n_3 \geqslant 0} \varepsilon_{n_1,n_2,n_3}\int_0^\infty H_{3\alpha}(t) \times \\ \left(\iint_{\substack{t_1,t_2\geqslant 0\\ t_1+t_2\leqslant t}} e^{-n_1a(\alpha+it_1)}e^{-n_2b(\alpha+it_2)}e^{-n_3c(\alpha+i(t-t_1-t_2))}dt_1dt_2\right)dt. \tag{2.8}$$

By

$$\iint_{\substack{t_1,t_2\geqslant 0\\ t_1+t_2\leqslant t}} \cdots dt_1dt_2 = \int_0^t\int_0^{t-t_1} \cdots dt_2dt_1 \qquad (t > 0)$$

and

$$n_1a = n_2b \Leftrightarrow (n_1, n_2) = (0, 0),\ \ n_2b = n_3c \Leftrightarrow (n_2, n_3) = (0, 0), \\ n_1a = n_3c \Leftrightarrow (n_1, n_3) = (0, 0), \tag{2.9}$$

which are introduced by Lemma 2.2, we calculate that

$$I_{333} = -\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3}e^{-(n_1a+n_2b+n_3c)\alpha} \times \\ \left(\int_0^\infty \frac{H_{3\alpha}(t)e^{-n_1ait}}{(n_1a - n_2b)(n_1a - n_3c)}dt + \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_2bit}}{(n_2b - n_1a)(n_2b - n_3c)}dt \right. \\ \left. + \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_3cit}}{(n_3c - n_1a)(n_3c - n_2b)}dt\right) \\ + \frac{abc}{32\pi^3}\sum_{n_1=1}^{\infty} e^{-n_1a\alpha}\left(\int_0^\infty \frac{tH_{3\alpha}(t)}{n_1ai}dt - \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_1ait}}{n_1^2a^2}dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_1^2a^2}dt\right)$$

$$
\begin{aligned}
&+\frac{abc}{32\pi^3}\sum_{n_2=1}^{\infty}e^{-n_2b\alpha}\left(\int_0^{\infty}\frac{tH_{3\alpha}(t)}{n_2bi}dt-\int_0^{\infty}\frac{H_{3\alpha}(t)e^{-n_2bit}}{n_2^2b^2}dt+\int_0^{\infty}\frac{H_{3\alpha}(t)}{n_2^2b^2}dt\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\int_0^{\infty}\frac{tH_{3\alpha}(t)}{n_3ci}dt-\int_0^{\infty}\frac{H_{3\alpha}(t)e^{-n_3cit}}{n_3^2c^2}dt+\int_0^{\infty}\frac{H_{3\alpha}(t)}{n_3^2c^2}dt\right)\\
&+\frac{abc}{128\pi^3}\int_0^{\infty}t^2H_{3\alpha}(t)dt. \qquad (2.10)
\end{aligned}
$$

where $\Gamma := \{(n_1,n_2,n_3)\in(\mathbb{Z}_{\geqslant 0})^3 : \text{the number of } i \text{ such that } n_i=0 \text{ is at most one}\}$.

Similarly

$$
\begin{aligned}
I_{111}=&-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_1a+n_2b+n_3c)\alpha}\times\\
&\left(\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_1ait}}{(n_1a-n_2b)(n_1a-n_3c)}dt+\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_2bit}}{(n_2b-n_1a)(n_2b-n_3c)}dt\right.\\
&\left.+\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_3cit}}{(n_3c-n_1a)(n_3c-n_2b)}dt\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_1=1}^{\infty}e^{-n_1a\alpha}\left(\int_{-\infty}^{0}\frac{tH_{3\alpha}(t)}{n_1ai}dt-\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_1ait}}{n_1^2a^2}dt+\int_{-\infty}^{0}\frac{H_{3\alpha}(t)}{n_1^2a^2}dt\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_2=1}^{\infty}e^{-n_2b\alpha}\left(\int_{-\infty}^{0}\frac{tH_{3\alpha}(t)}{n_2bi}dt-\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_2bit}}{n_2^2b^2}dt+\int_{-\infty}^{0}\frac{H_{3\alpha}(t)}{n_2^2b^2}dt\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\int_{-\infty}^{0}\frac{tH_{3\alpha}(t)}{n_3ci}dt-\int_{-\infty}^{0}\frac{H_{3\alpha}(t)e^{-n_3cit}}{n_3^2c^2}dt+\int_{-\infty}^{0}\frac{H_{3\alpha}(t)}{n_3^2c^2}dt\right)\\
&+\frac{abc}{128\pi^3}\int_{-\infty}^{0}t^2H_{3\alpha}(t)dt. \qquad (2.11)
\end{aligned}
$$

By (2.10) and (2.11) we have

$$
\begin{aligned}
&I_{111}+I_{333}\\
&=-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}e^{-(n_1a+n_2b+n_3c)\alpha}\times\\
&\left(\frac{\widetilde{H_{3\alpha}}(-n_1a)}{(n_1a-n_2b)(n_1a-n_3c)}+\frac{\widetilde{H_{3\alpha}}(-n_2b)}{(n_2b-n_1a)(n_2b-n_3c)}+\frac{\widetilde{H_{3\alpha}}(-n_3c)}{(n_3c-n_1a)(n_3c-n_2b)}\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_1=1}^{\infty}e^{-n_1a\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_1ai}-\frac{\widetilde{H_{3\alpha}}(-n_1a)}{n_1^2a^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_1^2a^2}\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_2=1}^{\infty}e^{-n_2b\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_2bi}-\frac{\widetilde{H_{3\alpha}}(-n_2b)}{n_2^2b^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_2^2b^2}\right)\\
&+\frac{abc}{32\pi^3}\sum_{n_3=1}^{\infty}e^{-n_3c\alpha}\left(\frac{\widetilde{tH_{3\alpha}(t)}(0)}{n_3ci}-\frac{\widetilde{H_{3\alpha}}(-n_3c)}{n_3^2c^2}+\frac{\widetilde{H_{3\alpha}}(0)}{n_3^2c^2}\right)\\
&+\frac{abc}{128\pi^3}\widetilde{t^2H_{3\alpha}(t)}(0)
\end{aligned}
$$

By Lemma 2.6 and $\widetilde{H}(-x)=\widetilde{H}(x)$, we have

$$
I_{111}+I_{333}
$$

$$
\begin{aligned}
&= -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1a+n_2b+n_3c)\alpha} \times \\
&\left(\frac{e^{3n_1a\alpha}\widetilde{H}(n_1a)}{(n_1a-n_2b)(n_1a-n_3c)} + \frac{e^{3n_2b\alpha}\widetilde{H}(n_2b)}{(n_2b-n_1a)(n_2b-n_3c)} + \frac{e^{3n_3c\alpha}\widetilde{H}(n_3c)}{(n_3c-n_1a)(n_3c-n_2b)} \right) \\
&+ \frac{abc}{32\pi^3} \left(3\alpha\widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1a} - \sum_{n_1=1}^{\infty} \frac{e^{2n_1a\alpha}\widetilde{H}(n_1a)}{n_1^2a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1^2a^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(3\alpha\widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2b} - \sum_{n_2=1}^{\infty} \frac{e^{2n_2b\alpha}\widetilde{H}(n_2b)}{n_2^2b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2^2b^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(3\alpha\widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3c} - \sum_{n_3=1}^{\infty} \frac{e^{2n_3c\alpha}\widetilde{H}(n_3c)}{n_3^2c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3^2c^2} \right) \\
&+ \frac{abc}{128\pi^3} \widetilde{t^2 H_{3\alpha}(t)}(0).
\end{aligned} \tag{2.12}
$$

Similarly we compute $I_{113}+I_{331}$, $I_{131}+I_{313}$ and $I_{311}+I_{133}$:

$$
\begin{aligned}
&I_{113}+I_{331} \\
&= -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1a+n_2b+n_3c)\alpha} \times \\
&\left(\frac{e^{-n_1a\alpha}\widetilde{H}(n_1a)}{(n_1a+n_2b)(n_1a+n_3c)} + \frac{e^{n_2b\alpha}\widetilde{H}(n_2b)}{(n_2b+n_1a)(n_2b-n_3c)} + \frac{e^{n_3c\alpha}\widetilde{H}(n_3c)}{(n_3c+n_1a)(n_3c-n_2b)} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1a} - \sum_{n_1=1}^{\infty} \frac{e^{-2n_1a\alpha}\widetilde{H}(n_1a)}{n_1^2a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1^2a^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2b} - \sum_{n_2=1}^{\infty} \frac{\widetilde{H}(n_2b)}{n_2^2b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2^2b^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3c} - \sum_{n_3=1}^{\infty} \frac{\widetilde{H}(n_3c)}{n_3^2c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3^2c^2} \right) \\
&+ \frac{abc}{128\pi^3} \widetilde{t^2 H_{\alpha}(t)}(0),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
&I_{131}+I_{313} \\
&= -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1a+n_2b+n_3c)\alpha} \times \\
&\left(\frac{e^{n_1a\alpha}\widetilde{H}(n_1a)}{(n_1a+n_2b)(n_1a-n_3c)} + \frac{e^{-n_2b\alpha}\widetilde{H}(n_2b)}{(n_2b+n_1a)(n_2b+n_3c)} + \frac{e^{n_3c\alpha}\widetilde{H}(n_3c)}{(n_3c-n_1a)(n_3c+n_2b)} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1a} - \sum_{n_1=1}^{\infty} \frac{\widetilde{H}(n_1a)}{n_1^2a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1a\alpha}}{n_1^2a^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2b} - \sum_{n_2=1}^{\infty} \frac{e^{-2n_2b\alpha}\widetilde{H}(n_2b)}{n_2^2b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2b\alpha}}{n_2^2b^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha\widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3c} - \sum_{n_3=1}^{\infty} \frac{\widetilde{H}(n_3c)}{n_3^2c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3c\alpha}}{n_3^2c^2} \right)
\end{aligned}
$$

$$+ \frac{abc}{128\pi^3}\widetilde{t^2 H_\alpha(t)}(0), \tag{2.14}$$

$$\begin{aligned}
&I_{311} + I_{133} \\
&= -\frac{abc}{8\pi^3} \sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\
&\quad \left(\frac{e^{n_1 a\alpha}\widetilde{H}(n_1 a)}{(n_1 a - n_2 b)(n_1 a + n_3 c)} + \frac{e^{n_2 b\alpha}\widetilde{H}(n_2 b)}{(n_2 b - n_1 a)(n_2 b + n_3 c)} + \frac{e^{-n_3 c\alpha}\widetilde{H}(n_3 c)}{(n_3 c + n_1 a)(n_3 c + n_2 b)} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1 a} - \sum_{n_1=1}^{\infty} \frac{\widetilde{H}(n_1 a)}{n_1^2 a^2} + \widetilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1^2 a^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2 b} - \sum_{n_2=1}^{\infty} \frac{\widetilde{H}(n_2 b)}{n_2^2 b^2} + \widetilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2^2 b^2} \right) \\
&+ \frac{abc}{32\pi^3} \left(\alpha \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3 c} - \sum_{n_3=1}^{\infty} \frac{e^{-2n_3 c\alpha}\widetilde{H}(n_3 c)}{n_3^2 c^2} + \widetilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2 c^2} \right) \\
&+ \frac{abc}{128\pi^3}\widetilde{t^2 H_\alpha(t)}(0).
\end{aligned} \tag{2.15}$$

Considering the limit $\alpha \to 0$ in the sum of (2.12), (2.13), (2.14) and (2.15), we have

$$\begin{aligned}
&\lim_{\alpha\to 0}(I_{111} + I_{113} + I_{131} + I_{133} + I_{311} + I_{313} + I_{331} + I_{333} + \frac{abc}{8\pi^3}(K_1(\alpha) + K_2(\alpha) + K_3(\alpha))) \\
&= -\frac{abc}{8\pi^3} \left(\sum_{n_1=1}^{\infty} \frac{\widetilde{H}(n_1 a)}{n_1^2 a^2} + \sum_{n_2=1}^{\infty} \frac{\widetilde{H}(n_2 b)}{n_2^2 b^2} + \sum_{n_3=1}^{\infty} \frac{\widetilde{H}(n_3 c)}{n_3^2 c^2} \right) \\
&+ \frac{abc}{8\pi^3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \zeta(2)\widetilde{H}(0) - \frac{abc}{32\pi^3}\widetilde{t^2 H(t)}(0),
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
K_1(\alpha) = &\sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_2 b + n_3 c)\alpha} \times \\
&\left(\frac{e^{n_1 a\alpha}(e^{n_1 a\alpha} + e^{-n_1 a\alpha})^2 n_1^2 a^2}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} + \frac{(e^{3n_1 a\alpha} - e^{-n_1 a\alpha}) n_1 n_2 ab}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \right. \\
&\left. + \frac{(e^{3n_1 a\alpha} - e^{-n_1 a\alpha}) n_1 n_3 ac}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} + \frac{e^{n_1 a\alpha}(e^{n_1 a\alpha} - e^{-n_1 a\alpha})^2 n_2 n_3 bc}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \right) \widetilde{H}(n_1 a), \\
K_2(\alpha) = &\sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1 a + n_3 c)\alpha} \times \\
&\left(\frac{e^{n_2 b\alpha}(e^{n_2 b\alpha} + e^{-n_2 b\alpha})^2 n_2^2 b^2}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} + \frac{(e^{3n_2 b\alpha} - e^{-n_2 b\alpha}) n_1 n_2 ab}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} \right. \\
&\left. + \frac{(e^{3n_2 b\alpha} - e^{-n_2 b\alpha}) n_2 n_3 bc}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} + \frac{e^{n_2 b\alpha}(e^{n_2 b\alpha} - e^{-n_2 b\alpha})^2 n_1 n_3 ac}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} \right) \widetilde{H}(n_2 b), \\
K_3(\alpha) = &\sum_{(n_1,n_2,n_3)\in\Gamma} \varepsilon_{n_1,n_2,n_3} e^{-(n_1 a + n_2 b)\alpha} \times
\end{aligned}$$

$$\left(\frac{e^{n_3c\alpha}(e^{n_3c\alpha}+e^{-n_3c\alpha})^2n_3^2c^2}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}+\frac{(e^{3n_3c\alpha}-e^{-n_3c\alpha})n_1n_3ac}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}\right.$$
$$\left.+\frac{(e^{3n_3c\alpha}-e^{-n_3c\alpha})n_2n_3bc}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}+\frac{e^{n_3c\alpha}(e^{n_3c\alpha}-e^{-n_3c\alpha})^2n_1n_2ab}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}\right)\widetilde{H}(n_3c),$$

and $\zeta(s)$ is the Riemann zeta function. Estimating $|K_j(\alpha)-K_j(0)|$ by using Lemma 2.5 and using $\widetilde{H}(x)\ll\mu^x$ for some $\mu\in(0,1)$, we have

$$\lim_{\alpha\to0}K_j(\alpha)=K_j(0)\tag{2.17}$$

for $j=1,2,3$. By (2.16) and (2.17), we have

$$\begin{aligned}
&\lim_{\alpha\to0}(I_{111}+I_{113}+I_{131}+I_{133}+I_{311}+I_{313}+I_{331}+I_{333})\\
&=-\frac{abc}{8\pi^3}\sum_{(n_1,n_2,n_3)\in\Gamma}\varepsilon_{n_1,n_2,n_3}\times\\
&\qquad\left(\frac{4n_1^2a^2}{(n_1^2a^2-n_2^2b^2)(n_1^2a^2-n_3^2c^2)}\widetilde{H}(n_1a)+\frac{4n_2^2b^2}{(n_2^2b^2-n_1^2a^2)(n_2^2b^2-n_3^2c^2)}\widetilde{H}(n_2b)\right.\\
&\qquad\left.+\frac{4n_3^2c^2}{(n_3^2c^2-n_1^2a^2)(n_3^2c^2-n_2^2b^2)}\widetilde{H}(n_3c)\right)\\
&\quad-\frac{abc}{8\pi^3}\left(\sum_{n_1=1}^{\infty}\frac{\widetilde{H}(n_1a)}{n_1^2a^2}+\sum_{n_2=1}^{\infty}\frac{\widetilde{H}(n_2b)}{n_2^2b^2}+\sum_{n_3=1}^{\infty}\frac{\widetilde{H}(n_3c)}{n_3^2c^2}\right)\\
&\quad+\frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0)-\frac{abc}{32\pi^3}\widetilde{t^2H(t)}(0).
\end{aligned}\tag{2.18}$$

By

$$\sum_{n>0}\frac{2ku}{k^2u^2-n^2v^2}+\frac{1}{ku}=\frac{\pi}{v}\cot\left(\pi\frac{ku}{v}\right),$$

(2.18) turns to

$$\begin{aligned}
&\lim_{\alpha\to0}(I_{111}+I_{113}+I_{131}+I_{133}+I_{311}+I_{313}+I_{331}+I_{333})\\
&=-\frac{a}{8\pi}\sum_{n_1>0}\cot\left(\pi\frac{n_1a}{b}\right)\cot\left(\pi\frac{n_1a}{c}\right)\widetilde{H}(n_1a)\\
&\quad-\frac{b}{8\pi}\sum_{n_2>0}\cot\left(\pi\frac{n_2b}{c}\right)\cot\left(\pi\frac{n_2b}{a}\right)\widetilde{H}(n_2b)\\
&\quad-\frac{c}{8\pi}\sum_{n_3>0}\cot\left(\pi\frac{n_3c}{a}\right)\cot\left(\pi\frac{n_3c}{b}\right)\widetilde{H}(n_3c)\\
&\quad+\frac{abc}{48\pi}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\widetilde{H}(0)-\frac{abc}{32\pi^3}\widetilde{H}''(0).
\end{aligned}\tag{2.19}$$

Next we calculate $I_{i_1i_2i_3}$ with $i_1=2,\ i_2=2$ or $i_3=2$. Put

$$I^{(1)}:=\sum_{j,k=1}^{3}I_{2jk},\quad I^{(2)}:=\sum_{j,k=1}^{3}I_{j2k},\quad I^{(3)}:=\sum_{j,k=1}^{3}I_{jk2},$$
$$I^{(4)}:=\sum_{j=1}^{3}I_{22j},\quad I^{(5)}:=\sum_{j=1}^{3}I_{2j2},\quad I^{(6)}:=\sum_{j=1}^{3}I_{j22}.$$

Then we have

$$
\begin{aligned}
I^{(1)} &= \frac{1}{(2\pi i)^3} \int_{C_2} \int_{\partial D} \int_{\partial D} h(s_1+s_2+s_3) \frac{Z_a'}{Z_a}(s_1) \frac{Z_b'}{Z_b}(s_2) \frac{Z_c'}{Z_c}(s_3) ds_1 ds_2 ds_3 \\
&= \frac{1}{2\pi i} \int_{C_2} \sum_{\rho_a, \rho_b} h(\rho_a+\rho_b+s_3) \frac{Z_c'}{Z_c}(s_3) ds_3 \\
&= \frac{1}{2\pi i} \int_{\pi}^{0} \sum_{\rho_a, \rho_b > 0} h(\rho_a+\rho_b+\alpha e^{i\theta}) \frac{Z_c'}{Z_c}(\alpha e^{i\theta}) \alpha i e^{i\theta} d\theta,
\end{aligned}
$$

where ρ_a, ρ_b run through the zeros of $Z_a(s), Z_b(s)$ with $\mathrm{Im}(\rho_a) > 0, \mathrm{Im}(\rho_b) > 0$, respectively. As $\alpha \to 0$, we have

$$
\lim_{\alpha \to 0} I^{(1)} = \frac{1}{2\pi i} \int_{\pi}^{0} \sum_{\rho_a, \rho_b} h(\rho_a+\rho_b) i d\theta = -\frac{1}{2} \sum_{\rho_a, \rho_b} h(\rho_a+\rho_b). \tag{2.20}
$$

We similarly deal with $I^{(k)} (k = 2, 3, \ldots, 6)$ and I_{222} to get

$$
\lim_{\alpha \to 0} I^{(2)} = -\frac{1}{2} \sum_{\rho_a, \rho_c} h(\rho_a+\rho_c), \tag{2.21}
$$

$$
\lim_{\alpha \to 0} I^{(3)} = -\frac{1}{2} \sum_{\rho_b, \rho_c} h(\rho_b+\rho_c). \tag{2.22}
$$

$$
\lim_{\alpha \to 0} I^{(4)} = \frac{1}{4} \sum_{\rho_a} h(\rho_a), \tag{2.23}
$$

$$
\lim_{\alpha \to 0} I^{(5)} = \frac{1}{4} \sum_{\rho_b} h(\rho_b), \tag{2.24}
$$

$$
\lim_{\alpha \to 0} I^{(6)} = \frac{1}{4} \sum_{\rho_a} h(\rho_a), \tag{2.25}
$$

$$
\lim_{\alpha \to 0} I_{222} = -\frac{1}{8} h(0), \tag{2.26}
$$

where ρ_k runs over the zeros of $Z_k(s)$ with $\mathrm{Im}(\rho_k) > 0$. By (2.20), (2.21), (2.22), (2.23), (2.24), (2.25) and (2.26), we have

$$
\begin{aligned}
&\lim_{\alpha \to 0} \sum_{(i_1, i_2, i_3) \in \{1,2,3\}^3 \setminus \{1,3\}^3} I_{i_1 i_2 i_3} \\
&= \lim_{\alpha \to 0} (I^{(1)} + I^{(2)} + I^{(3)} - (I^{(4)} + I^{(5)} + I^{(6)}) + I_{222}) \\
&= -\frac{1}{2} \left(\sum_{\rho_a, \rho_b} h(\rho_a+\rho_b) + \sum_{\rho_a, \rho_c} h(\rho_a+\rho_c) + \sum_{\rho_b, \rho_c > 0} h(\rho_b+\rho_c) \right) \\
&\quad - \frac{1}{4} \left(\sum_{\rho_a} h(\rho_a) + \sum_{\rho_b} h(\rho_b) + \sum_{\rho_c} h(\rho_c) \right) - \frac{1}{8} h(0).
\end{aligned} \tag{2.27}
$$

We apply (2.19) and (2.27) to the limit $\alpha \to 0$ in (2.5). This completes the proof. $\square$

3. Expression of Triple Sine

We define the multiple Hurwitz zeta function due to Barnes as

$$\zeta_r(s,z,\underline{\omega}) := \sum_{n_1,\ldots,n_r\geqslant 0} (n_1\omega_1+\cdots+n_r\omega_r+z)^{-s}$$

for $\underline{\omega}=(\omega_1,\ldots,\omega_r)\in(\mathbb{R}_{>0})^r$, the multiple gamma and the multiple sine as

$$\Gamma_r(z,\underline{\omega}) := \exp\left(\frac{\partial}{\partial s}\zeta_r(s,z,\underline{\omega})\Big|_{s=0}\right),$$
$$S_r(z,\underline{\omega}) := \Gamma_r(z,\underline{\omega})^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-z,\underline{\omega})^{(-1)^r}.$$

$S_3(z,\underline{\omega})$ has a following expression ([KuKo] Proposition 2.4):

$$S_3(z,\underline{\omega}) = e^{Q_{\underline{\omega}}(z)} z \prod_{n_1,n_2,n_3\geqslant 0}{}' P_3\left(-\frac{z}{n_1\omega_1+n_2\omega_2+n_3\omega_3}\right)\prod_{n_1,n_2,n_3\geqslant 1} P_3\left(\frac{z}{n_1\omega_1+n_2\omega_2+n_3\omega_3}\right), \quad (3.1)$$

where $P_r(u) := (1-u)\exp(u+\frac{u^2}{2}+\cdots+\frac{u^r}{r})$, and $Q_{\underline{\omega}}(z)\in\mathbb{C}[z]$ satisfies $\deg Q_{\underline{\omega}}(z)\leqslant 3$.

Lemma 3.1.

$$\frac{d^3}{dz^3}\log(1-e^{iaz}) = 2\sum_{n=-\infty}^{\infty}\frac{1}{(z-\frac{2\pi n}{a})^3}.$$

Proof. By

$$\log(1-e^{iaz}) = -\frac{\pi i}{2}+\frac{az}{2}i+\log\left(2\sin\frac{az}{2}\right)$$

and

$$2\sin\frac{az}{2} = az\prod_{n=1}^{\infty}\left(1-\left(\frac{az}{2\pi n}\right)^2\right)$$

we have

$$\begin{aligned}\frac{d^3}{dz^3}\log(1-e^{iaz}) &= \frac{2}{z^3}+2\sum_{n=1}^{\infty}\left(\frac{1}{(z-\frac{2\pi n}{a})^3}+\frac{1}{(z+\frac{2\pi n}{a})^3}\right)\\ &= 2\sum_{n=-\infty}^{\infty}\frac{1}{(z-\frac{2\pi n}{a})^3}.\end{aligned}$$

□

Lemma 3.2. *([KoKu] Theorem 2.3) Let $I(t)$ be odd function in $L^1(\mathbb{R})$ with satisfying following (i), (ii):*
(i)$I(t)=O(t^{-2})$ $(|t|\to\infty)$
(ii)There exists $\mu\in(0,1)$ such that $\widetilde{I}(x)=O(\mu^x)$ $(x\to\infty)$.
Let a,b be positive real numbers such that a/b is generic. Then we have

$$\begin{aligned}&\sum_{k,n>0} I\left(2\pi\left(\frac{k}{a}+\frac{n}{b}\right)\right)+\frac{1}{2}\left(\sum_{k>0} I\left(2\pi\frac{k}{a}\right)+\sum_{n>0} I\left(2\pi\frac{n}{b}\right)\right)\\ &= -\frac{ia}{4\pi}\sum_{k>0}\cot\left(\pi\frac{ka}{b}\right)\widetilde{I}(ka)-\frac{ib}{4\pi}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)\widetilde{I}(nb)-\frac{iab}{8\pi^2}\widetilde{I}'(0). \quad (3.2)\end{aligned}$$

Lemma 3.3. *([KoKu] Theorem 3.2) Assume ω_1/ω_2 is generic, then the double sine function has the following expression in* $\mathrm{Im}(z) > 0$*:*

$$\begin{aligned} S_2(z,(\omega_1,\omega_2)) = \exp\Bigg(& \frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi i k\frac{z}{\omega_1}} \\ & + \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi i n\frac{z}{\omega_2}} \\ & + \frac{1}{2}\log(1-e^{2\pi i\frac{z}{\omega_1}}) + \frac{1}{2}\log(1-e^{2\pi i\frac{z}{\omega_2}}) \\ & + \frac{\pi i z^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right)\Bigg). \end{aligned}$$

Theorem 3.4. *Let $\underline{\omega} = (\omega_1,\omega_2,\omega_3)$ be in $(\mathbb{R}_{>0})^3$ such that ω_i/ω_j are generic $(i,j=1,2,3; i\neq j)$. Then the triple sine function has the following expression in* $\mathrm{Im}(z) > 0$*:*

$$\begin{aligned} S_3(z,\underline{\omega}) = \exp\Bigg(& \frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)e^{2\pi i n_1\frac{z}{\omega_1}} \\ & + \frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}} \\ & + \frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}} \\ & + \frac{1}{4i}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)+\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right)e^{2\pi i n_1\frac{z}{\omega_1}} \\ & + \frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)e^{2\pi i n_2\frac{z}{\omega_2}} \\ & + \frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)+\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)\right)e^{2\pi i n_3\frac{z}{\omega_3}} \\ & + \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_1}}) + \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}}) \\ & + \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}}) - \frac{\pi i z^3}{6\omega_1\omega_2\omega_3} + \frac{\pi i}{4}\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_2\omega_3}+\frac{1}{\omega_3\omega_1}\right)z^2 \\ & - \frac{\pi i}{12}\left(\frac{3}{\omega_1}+\frac{3}{\omega_2}+\frac{3}{\omega_3}+\frac{\omega_1}{\omega_2\omega_3}+\frac{\omega_2}{\omega_3\omega_1}+\frac{\omega_3}{\omega_1\omega_2}\right)z \\ & + \frac{\pi i}{24}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+\frac{\omega_2}{\omega_3}+\frac{\omega_3}{\omega_2}+\frac{\omega_3}{\omega_1}+\frac{\omega_1}{\omega_3}+3\right)\Bigg) \end{aligned}$$

Proof. By (3.1) we have

$$\begin{aligned} & \frac{d^3}{dz^3}\left(\log S_3(z,\underline{\omega})\right) \\ & = C_{\underline{\omega}} + \frac{2}{z^3} \\ & +2\sum_{n_1,n_2,n_3\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_2\omega_2+n_3\omega_3))^3}+\frac{1}{(z-(n_1\omega_1+n_2\omega_2+n_3\omega_3))^3}\right) \end{aligned}$$

$$
\begin{aligned}
&+2\left(\sum_{n_1,n_2\geqslant 1}\frac{1}{(z+(n_1\omega_1+n_2\omega_2))^3}+\sum_{n_2,n_3\geqslant 1}\frac{1}{(z+(n_2\omega_2+n_3\omega_3))^3}\right.\\
&\qquad\left.+\sum_{n_1,n_3\geqslant 1}\frac{1}{(z+(n_1\omega_1+n_3\omega_3))^3}\right)\\
&+2\left(\sum_{n_1\geqslant 1}\frac{1}{(z+n_1\omega_1)^3}+\sum_{n_2\geqslant 1}\frac{1}{(z+n_2\omega_2)^2}+\sum_{n_3\geqslant 1}\frac{1}{(z+n_3\omega_3)^3}\right),
\end{aligned}
\tag{3.3}
$$

where $C_{\underline{\omega}}$ is a constant depending on $\underline{\omega}$.

Put

$$
H(t)=\frac{1}{(z+t)^3}+\frac{1}{(z-t)^3}\quad(\mathrm{Im}(z)>0).
$$

As we have

$$
\widetilde{H}(x)=2\pi i\,\mathrm{Res}_{t=z}(H(t)e^{itx})=\pi i x^2 e^{ixz}\quad(x\geqslant 0)
$$

and $H(t)=O(t^{-4})(|t|\to\infty)$, $H(t)$ satisfies (i) and (ii) in §2. As $\widetilde{H}''(0)=2\pi i$, by putting

$$
\begin{aligned}
F(z):=&\sum_{n_1,n_2,n_3\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_2\omega_2+n_3\omega_3))^3}+\frac{1}{(z-(n_1\omega_1+n_2\omega_2+n_3\omega_3))^3}\right)\\
&+\frac{1}{2}\sum_{n_1,n_2\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_2\omega_2))^3}+\frac{1}{(z-(n_1\omega_1+n_2\omega_2))^3}\right)\\
&+\frac{1}{2}\sum_{n_2,n_3\geqslant 1}\left(\frac{1}{(z+(n_2\omega_2+n_3\omega_3))^3}+\frac{1}{(z-(n_2\omega_2+n_3\omega_3))^3}\right)\\
&+\frac{1}{2}\sum_{n_1,n_3\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_3\omega_3))^3}+\frac{1}{(z-(n_1\omega_1+n_3\omega_3))^3}\right)\\
&+\frac{1}{4}\sum_{n_1\geqslant 1}\left(\frac{1}{(z+n_1\omega_1)^3}+\frac{1}{(z-n_1\omega_1)^3}\right)\\
&+\frac{1}{4}\sum_{n_2\geqslant 1}\left(\frac{1}{(z+n_2\omega_2)^3}+\frac{1}{(z-n_2\omega_2)^3}\right)\\
&+\frac{1}{4}\sum_{n_3\geqslant 1}\left(\frac{1}{(z+n_3\omega_3)^3}+\frac{1}{(z-n_3\omega_3)^3}\right)+\frac{1}{4z^3},
\end{aligned}
$$

the summation formula (2.3) shows

$$
\begin{aligned}
F(z)=&-\frac{a}{8}\sum_{n_1=1}^{\infty}\cot\left(\pi\frac{n_1a}{b}\right)\cot\left(\pi\frac{n_1a}{b}\right)i(n_1a)^2e^{in_1az}\\
&-\frac{b}{8}\sum_{n_2=1}^{\infty}\cot\left(\pi\frac{n_2b}{a}\right)\cot\left(\pi\frac{n_2b}{c}\right)i(n_2b)^2e^{in_2bz}\\
&-\frac{c}{8}\sum_{n_3=1}^{\infty}\cot\left(\pi\frac{n_3c}{a}\right)\cot\left(\pi\frac{n_3c}{b}\right)i(n_3c)^2e^{in_3cz}-\frac{abc}{16\pi^2}i\\
=&\frac{d^3}{dz^3}\left(\frac{1}{8}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1a}{b}\right)\cot\left(\pi\frac{n_1a}{c}\right)e^{in_1az}\right.
\end{aligned}
$$

$$+\frac{1}{8}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2b}{a}\right)\cot\left(\pi\frac{n_2b}{c}\right)e^{in_2bz}$$
$$+\frac{1}{8}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3c}{a}\right)\cot\left(\pi\frac{n_3c}{b}\right)e^{in_3cz}\Bigg)-\frac{abc}{16\pi^2}i$$

By (3.1) with $\omega_1=\frac{2\pi}{a}$, $\omega_2=\frac{2\pi}{b}$ and $\omega_3=\frac{2\pi}{c}$, we have

$$\begin{aligned}
&\frac{d^3}{dz^3}(\log S_3(z,\underline{\omega}))\\
&=C_{\underline{\omega}}+2F(z)+\frac{3}{2z^3}+\sum_{n_1,n_2\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_2\omega_2))^3}-\frac{1}{(z-(n_1\omega_1+n_2\omega_2))^3}\right)\\
&+\sum_{n_2,n_3\geqslant 1}\left(\frac{1}{(z+(n_2\omega_2+n_3\omega_3))^3}-\frac{1}{(z-(n_2\omega_2+n_3\omega_3))^3}\right)\\
&+\sum_{n_1,n_3\geqslant 1}\left(\frac{1}{(z+(n_1\omega_1+n_3\omega_3))^3}-\frac{1}{(z-(n_1\omega_1+n_3\omega_3))^3}\right)\\
&+\frac{3}{2}\left(\sum_{n_1\geqslant 1}\frac{1}{(z+n_1\omega_1)^3}+\sum_{n_2\geqslant 1}\frac{1}{(z+n_2\omega_2)^3}+\sum_{n_3\geqslant 1}\frac{1}{(z+n_3\omega_3)^3}\right)\\
&-\frac{1}{2}\left(\sum_{n_1\geqslant 1}\frac{1}{(z-n_1\omega_1)^3}+\sum_{n_2\geqslant 1}\frac{1}{(z-n_2\omega_2)^3}+\sum_{n_3\geqslant 1}\frac{1}{(z-n_3\omega_3)^3}\right).
\end{aligned}\tag{3.4}$$

Putting

$$I(t)=\frac{1}{(z+t)^3}-\frac{1}{(z-t)^3},$$

$I(t)$ is a odd function in $L^1(\mathbb{R})$ and satisties $I(t)=O(t^{-2})\quad(|t|\to\infty)$. Besides,

$$\widetilde{I}(x)=2\pi i\,\mathrm{Res}_{t=z}(I(t)e^{itx})=-\pi ix^2e^{ixz}\quad(x\geqslant 0).$$

Hence $I(t)$ satisfies the assumption of Lemma 3.2. By putting

$$G_1(z):=\sum_{n_1,n_2\geqslant 1}A\left(n_1\omega_1+n_2\omega_2\right)+\frac{1}{2}\left(\sum_{n_1\geqslant 1}A(n_1\omega_1)+\sum_{n_2\geqslant 1}A(n_2\omega_2)\right),$$
$$G_2(z):=\sum_{n_2,n_3\geqslant 1}A\left(n_2\omega_2+n_3\omega_3\right)+\frac{1}{2}\left(\sum_{n_2\geqslant 1}A(n_2\omega_2)+\sum_{n_3\geqslant 1}A(n_3\omega_3)\right),$$
$$G_3(z):=\sum_{n_1,n_3\geqslant 1}A\left(n_1\omega_1+n_3\omega_3\right)+\frac{1}{2}\left(\sum_{n_1\geqslant 1}A(n_1\omega_1)+\sum_{n_3\geqslant 1}A(n_3\omega_3)\right),$$

Lemma 3.2 shows

$$\begin{aligned}
G_1(z)&=\frac{1}{4i}\sum_{n_1=1}^{\infty}\cot\left(\pi\frac{n_1a}{b}\right)(in_1a)^3e^{in_1az}+\frac{1}{4i}\sum_{n_2=1}^{\infty}\cot\left(\pi\frac{n_2b}{a}\right)(in_2b)^3e^{in_2bz}\\
&=\frac{d^3}{dz^3}\left(\frac{1}{4i}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1a}{b}\right)e^{in_1az}+\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2b}{a}\right)e^{in_2bz}\right).
\end{aligned}\tag{3.5}$$

Similarly we compute

$$G_2(z) = \frac{d^3}{dz^3}\left(\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2 b}{c}\right)e^{in_2bz} + \frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3 c}{b}\right)e^{in_3cz}\right), \tag{3.6}$$

$$G_3(z) = \frac{d^3}{dz^3}\left(\frac{1}{4i}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1 a}{c}\right)e^{in_1az} + \frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3 c}{a}\right)e^{in_3cz}\right). \tag{3.7}$$

We express (3.4) using G_1, G_2 and G_3:

$$\begin{aligned}
&\frac{d^3}{dz^3}(\log S_3(z,\underline{\omega}))\\
&= C_{\underline{\omega}} + 2F(z) + G_1(z) + G_2(z) + G_3(z)\\
&+\frac{1}{2}\left(\sum_{n_1=-\infty}^{\infty}\frac{1}{(z-n_1\omega_1)^3} + \sum_{n_2=-\infty}^{\infty}\frac{1}{(z-n_2\omega_2)^3} + \sum_{n_3=-\infty}^{\infty}\frac{1}{(z-n_3\omega_3)^3}\right).
\end{aligned} \tag{3.8}$$

By (3.4), (3.5), (3.6), (3.7) and Lemma 3.1,

$$\begin{aligned}
Q(z) := \log S_3(z,\underline{\omega}) &- \frac{1}{4}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)e^{2\pi i n_1\frac{z}{\omega_1}}\\
&- \frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}\\
&- \frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}\\
&- \frac{1}{4i}\sum_{n_1=1}^{\infty}\frac{1}{n_1}\left(\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)+\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right)e^{2\pi i n_1\frac{z}{\omega_1}}\\
&- \frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)e^{2\pi i n_2\frac{z}{\omega_2}}\\
&- \frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)+\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)\right)e^{2\pi i n_3\frac{z}{\omega_3}}\\
&- \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_1}}) - \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}}) - \frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}})
\end{aligned} \tag{3.9}$$

is a polynomial of degree at most three. Thus we put $Q(z) = a + bz + cz^2 + dz^3$ and will compute a, b, c and d. First we compute b, c and d by considering

$$Q(z+\omega_1) - Q(z) = (b\omega_1 + c\omega_1^2 + d\omega_1^3) + (2c\omega_1 + 3d\omega_1^2)z + 3d\omega_1 z^2. \tag{3.10}$$

By (3.9) we have

$$\begin{aligned}
&Q(z+\omega_1) - Q(z)\\
&= \log\frac{S_3(z+\omega_1,\underline{\omega})}{S_3(z,\underline{\omega})}\\
&-\frac{1}{4}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)\\
&-\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)\\
&-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\left(\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)+\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)\\
&-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})\\
&+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}}).
\end{aligned}
\tag{3.11}
$$

Using the formula

$$
\cot(x)=i\frac{e^{2\pi i x}+1}{e^{2\pi i x}-1},
\tag{3.12}
$$

(3.11) turns to

$$
\begin{aligned}
&Q(z+\omega_1)-Q(z)\\
&=\log\frac{S_3(z+\omega_1,\underline{\omega})}{S_3(z,\underline{\omega})}\\
&-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)\\
&+\frac{1}{2i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}\\
&-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)\\
&+\frac{1}{2i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_2}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}\\
&-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})\\
&+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})+\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}}).
\end{aligned}
$$

Using the formula [KuKo, (2.4)] we get

$$
S_3(z+\omega_1,\underline{\omega})S_3(z,\underline{\omega})^{-1}=S_2(z,(\omega_2,\omega_3))^{-1}.
\tag{3.13}
$$

By Lemma 3.3 and (3.13), we have

$$
\begin{aligned}
&Q(z+\omega_1)-Q(z)\\
&=-\frac{1}{4i}\sum_{n_2=1}^{\infty}\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)e^{2\pi i n_2\frac{z}{\omega_2}}(e^{2\pi i n_2\frac{\omega_1}{\omega_2}}-1)\\
&-\frac{1}{4i}\sum_{n_3=1}^{\infty}\frac{1}{n_3}\cot\left(\pi\frac{n_3\omega_1}{\omega_3}\right)e^{2\pi i n_3\frac{z}{\omega_3}}(e^{2\pi i n_3\frac{\omega_1}{\omega_3}}-1)\\
&-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z+\omega_1}{\omega_3}})\\
&-\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_2}})-\frac{1}{4}\log(1-e^{2\pi i\frac{z}{\omega_3}})\\
&-\frac{\pi i z^2}{2\omega_2\omega_3}+\frac{\pi i}{2}\left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)z-\frac{\pi i}{12}\left(\frac{\omega_3}{\omega_2}+\frac{\omega_2}{\omega_3}+3\right).
\end{aligned}
\tag{3.14}
$$

By (3.12) we have

$$Q(z+\omega_1)-Q(z)=-\frac{\pi i z^2}{2\omega_2\omega_3}+\frac{\pi i}{2}\left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)z-\frac{\pi i}{12}\left(\frac{\omega_3}{\omega_2}+\frac{\omega_2}{\omega_3}+3\right). \tag{3.15}$$

Comparing the coefficients of (3.15), we have

$$\begin{aligned}
b&=-\frac{\pi i}{12}\left(\frac{3}{\omega_1}+\frac{3}{\omega_2}+\frac{3}{\omega_3}+\frac{\omega_1}{\omega_2\omega_3}+\frac{\omega_2}{\omega_3\omega_1}+\frac{\omega_3}{\omega_1\omega_2}\right),\\
c&=\frac{\pi i}{4}\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_2\omega_3}+\frac{1}{\omega_3\omega_1}\right),\\
d&=-\frac{\pi i}{6\omega_1\omega_2\omega_3}.
\end{aligned}$$

Next we will treat a by considering

$$\begin{aligned}
&Q(z)+Q(z+\frac{\omega_1}{2})+Q(z+\frac{\omega_2}{2})+Q(z+\frac{\omega_3}{2})+Q(z+\frac{\omega_1+\omega_2}{2})\\
&+Q(z+\frac{\omega_2+\omega_3}{2})+Q(z+\frac{\omega_3+\omega_1}{2})+Q(z+\frac{\omega_1+\omega_2+\omega_3}{2})\\
&-Q(2z).
\end{aligned} \tag{3.16}$$

The constant term of (3.16) is

$$7a-\frac{7\pi i}{24\omega_1\omega_2\omega_3}(\omega_1^2\omega_2+\omega_1\omega_2^2+\omega_2^2\omega_3+\omega_2\omega_3^2+\omega_3^2\omega_1+\omega_3\omega_1^2+3\omega_1\omega_2\omega_3). \tag{3.17}$$

On the other hand we will compute (3.16) by using (3.9). Putting

$$\begin{aligned}
A_0&:=\log\frac{\prod_{k_1,k_2,k_3=0}^{1}S_3\left(z+\dfrac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2},\underline{\omega}\right)}{S_3(2z,\underline{\omega})},\\
A_1&:=-\frac{1}{4}\sum_{n_1=1}^{\infty}\left\{\frac{1}{n_1}\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right.\\
&\qquad\left.\times\left(\sum_{k_1,k_2,k_3=0}^{1}e^{\frac{2\pi i n_1(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_1}}-e^{\frac{4\pi i n_1}{\omega_1}}\right)\right\}\\
&\quad-\frac{1}{4i}\sum_{n_1=1}^{\infty}\left\{\frac{1}{n_1}\left(\cot\left(\pi\frac{n_1\omega_2}{\omega_1}\right)+\cot\left(\pi\frac{n_1\omega_3}{\omega_1}\right)\right)\right.\\
&\qquad\left.\times\left(\sum_{k_1,k_2,k_3=0}^{1}e^{\frac{2\pi i n_1(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_1}}-e^{\frac{4\pi i n_1}{\omega_1}}\right)\right\},\\
A_2&:=-\frac{1}{4}\sum_{n_2=1}^{\infty}\left\{\frac{1}{n_2}\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right.\\
&\qquad\left.\times\left(\sum_{k_1,k_2,k_3=0}^{1}e^{\frac{2\pi i n_2(z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_2}}-e^{\frac{4\pi i n_2}{\omega_2}}\right)\right\}\\
&\quad-\frac{1}{4i}\sum_{n_2=1}^{\infty}\left\{\frac{1}{n_2}\left(\cot\left(\pi\frac{n_2\omega_1}{\omega_2}\right)+\cot\left(\pi\frac{n_2\omega_3}{\omega_2}\right)\right)\right.
\end{aligned}$$

$$\times \left(\sum_{k_1,k_2,k_3=0}^{1} e^{\frac{2\pi i n_2 (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_2}} - e^{\frac{4\pi i n_2}{\omega_2}} \right) \Bigg\},$$

$$A_3 := -\frac{1}{4} \sum_{n_3=1}^{\infty} \left\{ \frac{1}{n_3} \cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right) \right.$$

$$\left. \times \left(\sum_{k_1,k_2,k_3=0}^{1} e^{\frac{2\pi i n_3 (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_3}} - e^{\frac{4\pi i n_3}{\omega_3}} \right) \right\}$$

$$- \frac{1}{4i} \sum_{n_3=1}^{\infty} \left\{ \frac{1}{n_3} \left(\cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) + \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right) \right) \right.$$

$$\left. \times \left(\sum_{k_1,k_2,k_3=0}^{1} e^{\frac{2\pi i n_3 (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_3}} - e^{\frac{4\pi i n_3}{\omega_3}} \right) \right\},$$

$$A_4 := -\frac{1}{4} \log \frac{\prod_{k_1,k_2,k_3=0}^{1} \left(1 - e^{\frac{2\pi i (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_1}} \right)}{1 - e^{\frac{4\pi i z}{\omega_1}}},$$

$$A_5 := -\frac{1}{4} \log \frac{\prod_{k_1,k_2,k_3=0}^{1} \left(1 - e^{\frac{2\pi i (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_2}} \right)}{1 - e^{\frac{4\pi i z}{\omega_2}}},$$

$$A_6 := -\frac{1}{4} \log \frac{\prod_{k_1,k_2,k_3=0}^{1} \left(1 - e^{\frac{2\pi i (z+\frac{k_1\omega_1+k_2\omega_2+k_3\omega_3}{2})}{\omega_3}} \right)}{1 - e^{\frac{4\pi i z}{\omega_3}}},$$

we write (3.16) as $\sum_{j=0}^{6} A_j$. The formula [KuKo, (2.5)] gives $A_0 = 0$. Computing A_1 by (3.12), we have

$$A_1 = -\frac{1}{4} \sum_{n_1=1}^{\infty} \left\{ \frac{1}{n_1} (e^{\frac{4\pi i n_1 (z+\frac{\omega_2+\omega_3}{2})}{\omega_1}} + e^{\frac{4\pi i n_1 (z+\frac{\omega_2}{2})}{\omega_1}} + e^{\frac{4\pi i n_1 (z+\frac{\omega_3}{2})}{\omega_1}}) \right\}$$

$$= \frac{1}{4} \log\{(1 - e^{\frac{4\pi i (z+\frac{\omega_2+\omega_3}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z+\frac{\omega_2}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z+\frac{\omega_3}{2})}{\omega_1}})\}.$$

A_4 is easily computed as

$$A_4 = -\frac{1}{4} \log\{(1 - e^{\frac{4\pi i (z+\frac{\omega_2+\omega_3}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z+\frac{\omega_2}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z+\frac{\omega_3}{2})}{\omega_1}})\}.$$

Therefore $A_1 + A_4 = 0$. Similarly computing we have $A_2 + A_5 = A_3 + A_6 = 0$. Hence $\sum_{j=0}^{6} A_j = 0$. Therefore its constant term (3.17) vanishes, which lead to

$$a = \frac{\pi i}{24} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} + 3 \right).$$

□

4. Proof of Theorem 1.2

Let $Z_j (j=1,\dots,r)$ be meromorphic functions of order μ_j. We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod_{\rho\in\mathbb{C}}{}' P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)},$$

where $P_r(u) := (1-u)\exp(u+\frac{u^2}{2}+\cdots+\frac{u^r}{r})$, m_j denotes the multiplicity function for Z_j, $k_j := m_j(0)$ and Q_j is a polynomial with $\deg Q_j(s) \leqslant \mu_j$. The product $\prod_{\rho\in\mathbb{C}}{}'$ means $\lim_{R\to\infty}\prod_{0<|\rho|<R}$. Then, we have

$$\begin{aligned}Z_1(s)\otimes\cdots\otimes Z_r(s) &= s^{k_1\cdots k_r} e^{Q(s)} \\ &\times \prod_{\rho_1,\dots,\rho_r\in\mathbb{C}}{}' P_{\mu_1+\cdots+\mu_r}\left(\frac{s}{\rho_1+\cdots+\rho_r}\right)^{m(\rho_1,\cdots,\rho_r)},\end{aligned}$$

where $Q(s)$ is a polynomial with $\deg Q(s) \leqslant \mu_1+\cdots+\mu_r$ and

$$m(\rho_1,\dots,\rho_r) := m_1(\rho_1)\cdots m_r(\rho_r)\times\begin{cases}1 & \text{if } \mathrm{Im}(\rho_1),\dots,\mathrm{Im}(\rho_r)\geqslant 0,\\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_1),\dots,\mathrm{Im}(\rho_r)<0,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.1. *The absolute tensor product of Hasse zeta function for finite fields is given as follows:*

$$\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r) = e^{Q(s)} S_3\left(is,\left(\frac{2\pi}{\log p},\frac{2\pi}{\log q},\frac{2\pi}{\log r}\right)\right)^{-1},$$

where p,q,r are powers of prime and $Q(s)\in\mathbb{C}[s]$ satisfies $\deg Q(s)\leqslant 3$ depending on p,q,r.

Proof. We compute that the Hadamard product for Hasse zeta function is given by

$$\zeta(s,\mathbb{F}_p) = s^{-1} e^{\tilde{Q}_p(s)} \prod_{n=-\infty}^{\infty}{}' P_1\left(\frac{s}{\frac{2\pi i}{\log p}n}\right)^{-1},$$

where $\tilde{Q}_j(s)$ is a linear polynomial depending on p. Thus by the definition of the absolute tensor product,

$$\begin{aligned}\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r) &= s^{-1} e^{\tilde{Q}_{p,q,r}(s)}\times\\ \prod_{n_1,n_2,n_3\in\mathbb{Z}}{}' P_3&\left(\frac{s}{\frac{2\pi i}{\log p}n_1+\frac{2\pi i}{\log q}n_2+\frac{2\pi i}{\log r}n_3}\right)^{m_{n_1,n_2,n_3}},\end{aligned}$$

where $\tilde{Q}_{p,q,r}(s)\in\mathbb{C}[s]$ satisfies $\deg\tilde{Q}_{p,q,r}(s)\leqslant 3$ and

$$\begin{aligned}m_{n_1,n_2,n_3} &= m\left(\frac{2\pi i}{\log p}n_1,\frac{2\pi i}{\log q}n_2,\frac{2\pi i}{\log r}n_3\right)\\ &= \begin{cases}-1 & \text{if } n_1,n_2,n_3\geqslant 0 \text{ or } n_1,n_2,n_3<0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}&\zeta(s,\mathbb{F}_p)\otimes\zeta(s,\mathbb{F}_q)\otimes\zeta(s,\mathbb{F}_r)\\ &= s^{-1} e^{\tilde{Q}_{p,q,r}(s)} \prod_{n_1,n_2,n_3\geqslant 0}{}' P_3\left(\frac{s}{\frac{2\pi i}{\log p}n_1+\frac{2\pi i}{\log q}n_2+\frac{2\pi i}{\log r}n_3}\right)^{-1}\times\end{aligned}$$

$$\prod_{n_1,n_2,n_3 \geqslant 1} P_3\left(-\frac{s}{\frac{2\pi i}{\log p}n_1 + \frac{2\pi i}{\log q}n_2 + \frac{2\pi i}{\log r}n_3}\right)^{-1}.$$

By (3.1) the result follows. □

Proof of Theorem 1.2. Applying Theorem 3.4 with $z = is$ and $(\omega_1, \omega_2, \omega_3) = (\frac{2\pi}{\log p}, \frac{2\pi}{\log q}, \frac{2\pi}{\log r})$, Theorem 1.2 follows. □

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