

# Euler Product Expression of Triple Zeta Functions

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## ABSTRACT

We construct multiple zeta functions considered as absolute tensor products of usual zeta functions. We establish Euler product expressions for triple zeta functions  $\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r)$  with  $p, q, r$  distinct primes, via multiple sine functions by using the signatured Poisson summation formula.

## 1. Introduction and Main Theorem

Let

$$Z_j(s) = \prod_{\rho \in \mathbb{C}} (s - \rho)^{m_j(\rho)}$$

be “zeta functions” expressed as regularized products in the notation of Deninger [D1], [D2], where  $m_j : \mathbb{C} \rightarrow \mathbb{Z}$  denotes the multiplicity function for  $j = 1, \dots, r$ . (Later we will specify “zeta functions” to be treated.) Then, as in the paper [K] we define the absolute tensor product  $Z_1(s) \otimes \dots \otimes Z_r(s)$  as

$$Z_1(s) \otimes \dots \otimes Z_r(s) = \prod_{\rho_1, \dots, \rho_r \in \mathbb{C}} (s - (\rho_1 + \dots + \rho_r))^{m(\rho_1, \dots, \rho_r)},$$

where

$$m(\rho_1, \dots, \rho_r) := m_1(\rho_1) \cdots m_r(\rho_r) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \dots, \operatorname{Im}(\rho_r) \geq 0, \\ (-1)^{r-1} & \text{if } \operatorname{Im}(\rho_1), \dots, \operatorname{Im}(\rho_r) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define Hasse zeta functions  $\zeta(s, A)$  for commutative rings  $A$  as

$$\zeta(s, A) := \prod_{\mathfrak{m}} (1 - N(\mathfrak{m})^{-s})^{-1},$$

where  $\mathfrak{m}$  runs over maximal ideals of  $A$ , and  $N(\mathfrak{m}) := \#(A/\mathfrak{m})$ .

The following result about the absolute tensor product of Hasse zeta functions is known:

**THEOREM 1.1.** ([KoKu] Theorem 1.1) *The following expressions hold in  $\operatorname{Re}(s) > 0$  with some polynomial  $Q(s)$  of degree at most two:*

(1) *When  $p \neq q$ , it holds that*

$$\begin{aligned} \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) &= e^{Q_{p,q}(s)} (1 - p^{-s})^{\frac{1}{2}} (1 - q^{-s})^{\frac{1}{2}} \times \\ &\exp \left( \frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot(\pi k \log p)}{k} p^{-ks} + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot(\pi n \log q)}{n} q^{-ns} \right). \end{aligned}$$

(2) When  $p = q$ , it holds that

$$\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_p) = e^{Q_{p,p}(s)} (1 - p^{-s})^{1 - \frac{is \log p}{2\pi}} \exp\left(-\frac{\text{Li}_2(p^{-s})}{2\pi i}\right),$$

where

$$\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

In this paper we treat  $\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r)$  when  $p, q, r$  are distinct primes. We prove the following theorem, which is similar to Theorem 1.1(1):

**THEOREM 1.2.** *Suppose that  $p, q, r$  are distinct primes and  $s \in \mathbb{C}$  satisfies  $\text{Re}(s) > 0$ . Then,*

$$\begin{aligned} & \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r) \\ &= e^{Q_{p,q,r}(s)} (1 - p^{-s})^{-\frac{1}{4}} (1 - q^{-s})^{-\frac{1}{4}} (1 - r^{-s})^{-\frac{1}{4}} \\ & \quad \exp\left(-\frac{1}{4} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi n_1 \frac{\log p}{\log q}\right) \cot\left(\pi n_1 \frac{\log p}{\log r}\right) p^{-n_1 s}\right. \\ & \quad - \frac{1}{4} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi n_2 \frac{\log q}{\log p}\right) \cot\left(\pi n_2 \frac{\log q}{\log r}\right) q^{-n_2 s} \\ & \quad - \frac{1}{4} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi n_3 \frac{\log r}{\log p}\right) \cot\left(\pi n_3 \frac{\log r}{\log q}\right) r^{-n_3 s} \\ & \quad + \frac{i}{4} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \left( \cot\left(\pi n_1 \frac{\log p}{\log q}\right) + \cot\left(\pi n_1 \frac{\log p}{\log r}\right) \right) p^{-n_1 s} \\ & \quad + \frac{i}{4} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \left( \cot\left(\pi n_2 \frac{\log q}{\log p}\right) + \cot\left(\pi n_2 \frac{\log q}{\log r}\right) \right) q^{-n_2 s} \\ & \quad \left. + \frac{i}{4} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \left( \cot\left(\pi n_3 \frac{\log r}{\log p}\right) + \cot\left(\pi n_3 \frac{\log r}{\log q}\right) \right) r^{-n_3 s} \right), \end{aligned}$$

where  $Q_{p,q,r}(s)$  is a polynomial of degree at most three, which depends on  $p, q, r$ .

## 2. Triple Poisson Summation Formula with Signature

**DEFINITION 2.1.**  $\alpha \in \mathbb{R}$  is said to be *generic*, if

$$\lim_{m \rightarrow \infty} \|m\alpha\|^{\frac{1}{m}} = 1,$$

where we put  $\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}$  for  $x \in \mathbb{R}$ .

**LEMMA 2.2.** *If  $\alpha \in \mathbb{Q}$ , then  $\alpha$  is not generic.*

*Proof.* Let  $\alpha = a/b$  with  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}_{>0}$ . Then  $\|(bm)\alpha\| = 0$  for all  $m \in \mathbb{Z}$ . Hence  $\alpha$  is not generic.  $\square$

**LEMMA 2.3.** (Baker, [B]Theorem 3.1) *Let  $\alpha$  and  $\beta$  be in  $\overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$ . If  $\frac{\log \alpha}{\log \beta} \notin \mathbb{Q}$ , then  $\frac{\log \alpha}{\log \beta}$  is generic.*

**LEMMA 2.4.** *Let  $\alpha$  and  $\beta$  be generic. Then*

$$\sum_{n=1}^{\infty} \cot(\pi n\alpha) \cot(\pi n\beta) x^n \tag{2.1}$$

converges absolutely in  $|x| < 1$ .

*Proof.* Since  $\alpha, \beta$  are generic,  $\|n\alpha\|^{-1}, \|n\beta\|^{-1} = O(e^{\varepsilon n})$  as  $n \rightarrow \infty$  for any  $\varepsilon > 0$ . Since  $\cot(\pi x) \sim 1/(\pi x)$  ( $x \rightarrow 0$ ),  $\cot(\pi n\alpha) \cot(\pi n\beta) = O(e^{2\varepsilon n})$  ( $n \rightarrow \infty$ ). Hence the radius of convergence of (2.1), say  $R$ , satisfies

$$R = \limsup_{n \rightarrow \infty} \left| \frac{1}{\cot(\pi n\alpha) \cot(\pi n\beta)} \right|^{\frac{1}{n}} \geq e^{-2\varepsilon}.$$

Since  $\varepsilon > 0$  is arbitrary,  $R \geq 1$ .  $\square$

LEMMA 2.5. Let  $n, m$  be positive integers and  $a, b$  be positive real numbers such that  $a/b$  is generic. Then, for any  $\varepsilon > 0$  we have

$$|na - mb|^{-1} \ll_{a,b,\varepsilon} m^{-1} e^{\varepsilon n}$$

*Proof.* For all  $n, m \in \mathbb{Z}_{>0}$  we have

$$|na - mb| = mb \left| \frac{na}{mb} - 1 \right| \geq mb \left( \left[ \frac{na}{b} \right] + 1 \right)^{-1} \left\| \frac{na}{b} \right\|,$$

where  $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$ . Since  $a/b$  is generic, for any  $\varepsilon > 0$  we have

$$\left\| n \frac{a}{b} \right\|^{-1} \ll_{a,b,\varepsilon} e^{\varepsilon n}.$$

These show the lemma.  $\square$

For  $f \in L^1(\mathbb{R})$ , we denote by  $\tilde{f}$  the Fourier transform of  $f$ :

$$\tilde{f}(x) := \int_{-\infty}^{\infty} f(t) e^{itx} dt.$$

Let  $H(t)$  be an even regular function on  $\{z = x + iy : x \in \mathbb{R}, y \in (-R, R)\}$ , which satisfies (i) and (ii):

(i) There exists  $\delta > 0$  such that  $H(t) = O(t^{-3-\delta})$  ( $|t| \rightarrow \infty$ )

(ii) There exists  $\mu \in (0, 1)$  such that  $\tilde{H}(x) = O(\mu^x)$  ( $x \rightarrow \infty$ )

$h(t)$  and  $H_\alpha(t)$  denote  $h(t) := H(t/i)$ ,  $H_\alpha(t) := h(\alpha + it)$ , respectively.

LEMMA 2.6. Let  $\alpha \in (-R, R)$  and  $x \in \mathbb{R}$ . Then,

$$(1) \widetilde{H_\alpha}(x) = e^{-\alpha x} \tilde{H}(x).$$

$$(2) \widetilde{tH_\alpha}(t)(0) = i\alpha \tilde{H}(0).$$

*Proof.* (1) Applying Cauchy's theorem to  $H(t)e^{itx}$ , we have

$$\int_{C_T} H(t) e^{itx} dt = 0,$$

where

$$C_T := \partial\{z \in \mathbb{C} \mid -T < \operatorname{Re}(z) < T, \min\{-\alpha, 0\} < \operatorname{Im}(z) < \max\{-\alpha, 0\}\}.$$

Considering the limit  $T \rightarrow \infty$ , (1) follows.

(2) Considering the same as (1), we have

$$\int_{-\infty}^{\infty} tH(t) dt = \int_{-\infty}^{\infty} (t - i\alpha) H_\alpha(t) dt. \quad (2.2)$$

Since  $tH(t)$  is an odd function, left side of (2.2) equals to 0. So

$$\widetilde{tH_\alpha}(t)(0) = i\alpha \tilde{H}_\alpha(0)$$

$$= i\alpha \tilde{H}(0).$$

□

**THEOREM 2.7.** *Let  $a, b, c \in \mathbb{R}$  be positive real numbers such that  $a/b, b/c, c/a$  and their inverses are generic. Then we have*

$$\begin{aligned}
& \sum_{n_1, n_2, n_3 > 0} H\left(2\pi\left(\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c}\right)\right) + \frac{1}{2} \left( \sum_{n_1, n_2 > 0} H\left(2\pi\left(\frac{n_1}{a} + \frac{n_2}{b}\right)\right) \right. \\
& \quad \left. + \sum_{n_2, n_3 > 0} H\left(2\pi\left(\frac{n_2}{b} + \frac{n_3}{c}\right)\right) + \sum_{n_1, n_3 > 0} H\left(2\pi\left(\frac{n_1}{a} + \frac{n_3}{c}\right)\right) \right) \\
& \quad + \frac{1}{4} \left( \sum_{n_1 > 0} H\left(2\pi\frac{n_1}{a}\right) + \sum_{n_2 > 0} H\left(2\pi\frac{n_2}{b}\right) + \sum_{n_3 > 0} H\left(2\pi\frac{n_3}{c}\right) \right) + \frac{1}{8} H(0) \\
& = -\frac{a}{8\pi} \sum_{n_1 > 0} \cot\left(\pi\frac{n_1 a}{b}\right) \cot\left(\pi\frac{n_1 a}{c}\right) \tilde{H}(n_1 a) \\
& \quad - \frac{b}{8\pi} \sum_{n_2 > 0} \cot\left(\pi\frac{n_2 b}{c}\right) \cot\left(\pi\frac{n_2 b}{a}\right) \tilde{H}(n_2 b) \\
& \quad - \frac{c}{8\pi} \sum_{n_3 > 0} \cot\left(\pi\frac{n_3 c}{a}\right) \cot\left(\pi\frac{n_3 c}{b}\right) \tilde{H}(n_3 c) \\
& \quad + \frac{abc}{48\pi} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \tilde{H}(0) - \frac{abc}{32\pi^3} \tilde{H}''(0). \tag{2.3}
\end{aligned}$$

*Proof.* Put  $Z_k(s) = \sinh\left(\frac{ks}{2}\right)$  ( $k = a, b, c$ ). Let  $D_T$  be the region defined by

$$D_T := \{s \in \mathbb{C} : |s| > \alpha, |\operatorname{Re}(s)| < \alpha, 0 < \operatorname{Im}(s) < T\},$$

$$\text{where } 0 < \alpha < \min\left\{\frac{2\pi}{a}, \frac{2\pi}{b}, \frac{2\pi}{c}\right\}.$$

By Cauchy's theorem we have

$$\begin{aligned}
& \sum_{0 < \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2), \operatorname{Im}(\rho_3) < T} h(\rho_a + \rho_b + \rho_c) \\
& = \frac{1}{(2\pi i)^3} \int_{\partial D_T} \int_{\partial D_T} \int_{\partial D_T} h(s_1 + s_2 + s_3) \frac{Z'_a}{Z_a}(s_1) \frac{Z'_b}{Z_b}(s_2) \frac{Z'_c}{Z_c}(s_3) ds_1 ds_2 ds_3, \tag{2.4}
\end{aligned}$$

where  $\rho_k$  denotes the zeros of  $Z_k(s)$  ( $k = a, b, c$ ), and the contour  $\partial D_T$  is taken counterclockwise. Considering  $T \rightarrow \infty$  in (2.4), we have

$$\begin{aligned}
& \sum_{0 < \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2), \operatorname{Im}(\rho_3)} h(\rho_1 + \rho_2 + \rho_3) \\
& = \frac{1}{(2\pi i)^3} \int_{\partial D} \int_{\partial D} \int_{\partial D} h(s_1 + s_2 + s_3) \frac{Z'_a}{Z_a}(s_1) \frac{Z'_b}{Z_b}(s_2) \frac{Z'_c}{Z_c}(s_3) ds_1 ds_2 ds_3, \tag{2.5}
\end{aligned}$$

where

$$D := \{s \in \mathbb{C} : |\operatorname{Re}(s)| < \alpha, |s| > \alpha, \operatorname{Im}(s) > 0\}.$$

We decompose  $\partial D = C_1 \cup C_2 \cup C_3$  with

$$\begin{aligned}
C_1 &:= \{s \in \partial D : \operatorname{Re}(s) = -\alpha\}, \\
C_2 &:= \{s \in \partial D : |s| = \alpha\},
\end{aligned}$$

$$C_3 := \{s \in \partial D : \operatorname{Re}(s) = \alpha\}.$$

We compute each triple integral  $I_{i_1 i_2 i_3} = \int_{C_{i_1}} \int_{C_{i_2}} \int_{C_{i_3}}$  in (2.5).

First we calculate  $I_{i_1 i_2 i_3}$  with  $(i_1, i_2, i_3) \in \{1, 3\}^3$ .

$$\begin{aligned} I_{333} &= \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\infty \int_0^\infty h(3\alpha + i(t_1 + t_2 + t_3)) \\ &\quad \frac{Z'_a}{Z_a}(\alpha + it_1) \frac{Z'_b}{Z_b}(\alpha + it_2) \frac{Z'_c}{Z_c}(\alpha + it_3) dt_1 dt_2 dt_3. \end{aligned} \quad (2.6)$$

Since

$$\frac{Z'_k}{Z_k}(s) = \frac{k}{2} + k \sum_{n=1}^{\infty} e^{-kns} \quad (2.7)$$

for  $k = a, b, c$  and  $\operatorname{Re}(s) > 0$ , (2.6) turns to

$$\begin{aligned} I_{333} &= \sum_{n_1, n_2, n_3 \geq 0} \varepsilon_{n_1, n_2, n_3} \int_0^\infty \int_0^\infty \int_0^\infty H_{3\alpha}(t_1 + t_2 + t_3) \\ &\quad e^{-n_1 a(\alpha + it_1)} e^{-n_2 b(\alpha + it_2)} e^{-n_3 c(\alpha + it_3)} dt_1 dt_2 dt_3, \end{aligned}$$

where

$$\varepsilon_{n_1, n_2, n_3} := \left(\frac{1}{2}\right)^{\#\{i \in \{1, 2, 3\} : n_i = 0\}}.$$

We replace  $t_3$  with  $t = t_1 + t_2 + t_3$  to get

$$\begin{aligned} I_{333} &= \frac{1}{8\pi^3} \sum_{n_1, n_2, n_3 \geq 0} \varepsilon_{n_1, n_2, n_3} \int_0^\infty H_{3\alpha}(t) \times \\ &\quad \left( \iint_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 \leq t}} e^{-n_1 a(\alpha + it_1)} e^{-n_2 b(\alpha + it_2)} e^{-n_3 c(\alpha + i(t - t_1 - t_2))} dt_1 dt_2 \right) dt. \end{aligned} \quad (2.8)$$

By

$$\iint_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 \leq t}} \cdots dt_1 dt_2 = \int_0^t \int_0^{t-t_1} \cdots dt_2 dt_1 \quad (t > 0)$$

and

$$\begin{aligned} n_1 a = n_2 b &\Leftrightarrow (n_1, n_2) = (0, 0), \quad n_2 b = n_3 c \Leftrightarrow (n_2, n_3) = (0, 0), \\ n_1 a = n_3 c &\Leftrightarrow (n_1, n_3) = (0, 0), \end{aligned} \quad (2.9)$$

which are introduced by Lemma 2.2, we calculate that

$$\begin{aligned} I_{333} &= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\ &\quad \left( \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_1 ait}}{(n_1 a - n_2 b)(n_1 a - n_3 c)} dt + \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_2 bit}}{(n_2 b - n_1 a)(n_2 b - n_3 c)} dt \right. \\ &\quad \left. + \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_3 cit}}{(n_3 c - n_1 a)(n_3 c - n_2 b)} dt \right) \\ &\quad + \frac{abc}{32\pi^3} \sum_{n_1=1}^{\infty} e^{-n_1 a\alpha} \left( \int_0^\infty \frac{tH_{3\alpha}(t)}{n_1 ai} dt - \int_0^\infty \frac{H_{3\alpha}(t)e^{-n_1 ait}}{n_1^2 a^2} dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_1^2 a^2} dt \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{abc}{32\pi^3} \sum_{n_2=1}^{\infty} e^{-n_2 b \alpha} \left( \int_0^\infty \frac{t H_{3\alpha}(t)}{n_2 b i} dt - \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_2 b i t}}{n_2^2 b^2} dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_2^2 b^2} dt \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_3=1}^{\infty} e^{-n_3 c \alpha} \left( \int_0^\infty \frac{t H_{3\alpha}(t)}{n_3 c i} dt - \int_0^\infty \frac{H_{3\alpha}(t) e^{-n_3 c i t}}{n_3^2 c^2} dt + \int_0^\infty \frac{H_{3\alpha}(t)}{n_3^2 c^2} dt \right) \\
 & + \frac{abc}{128\pi^3} \int_0^\infty t^2 H_{3\alpha}(t) dt.
 \end{aligned} \tag{2.10}$$

where  $\Gamma := \{(n_1, n_2, n_3) \in (\mathbb{Z}_{\geq 0})^3 : \text{the number of } i \text{ such that } n_i = 0 \text{ is at most one}\}$ .

Similarly

$$\begin{aligned}
 I_{111} = & -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c) \alpha} \times \\
 & \left( \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_1 a i t}}{(n_1 a - n_2 b)(n_1 a - n_3 c)} dt + \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_2 b i t}}{(n_2 b - n_1 a)(n_2 b - n_3 c)} dt \right. \\
 & \left. + \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_3 c i t}}{(n_3 c - n_1 a)(n_3 c - n_2 b)} dt \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_1=1}^{\infty} e^{-n_1 a \alpha} \left( \int_{-\infty}^0 \frac{t H_{3\alpha}(t)}{n_1 a i} dt - \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_1 a i t}}{n_1^2 a^2} dt + \int_{-\infty}^0 \frac{H_{3\alpha}(t)}{n_1^2 a^2} dt \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_2=1}^{\infty} e^{-n_2 b \alpha} \left( \int_{-\infty}^0 \frac{t H_{3\alpha}(t)}{n_2 b i} dt - \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_2 b i t}}{n_2^2 b^2} dt + \int_{-\infty}^0 \frac{H_{3\alpha}(t)}{n_2^2 b^2} dt \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_3=1}^{\infty} e^{-n_3 c \alpha} \left( \int_{-\infty}^0 \frac{t H_{3\alpha}(t)}{n_3 c i} dt - \int_{-\infty}^0 \frac{H_{3\alpha}(t) e^{-n_3 c i t}}{n_3^2 c^2} dt + \int_{-\infty}^0 \frac{H_{3\alpha}(t)}{n_3^2 c^2} dt \right) \\
 & + \frac{abc}{128\pi^3} \int_{-\infty}^0 t^2 H_{3\alpha}(t) dt.
 \end{aligned} \tag{2.11}$$

By (2.10) and (2.11) we have

$$\begin{aligned}
 & I_{111} + I_{333} \\
 = & -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c) \alpha} \times \\
 & \left( \frac{\widetilde{H}_{3\alpha}(-n_1 a)}{(n_1 a - n_2 b)(n_1 a - n_3 c)} + \frac{\widetilde{H}_{3\alpha}(-n_2 b)}{(n_2 b - n_1 a)(n_2 b - n_3 c)} + \frac{\widetilde{H}_{3\alpha}(-n_3 c)}{(n_3 c - n_1 a)(n_3 c - n_2 b)} \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_1=1}^{\infty} e^{-n_1 a \alpha} \left( \frac{\widetilde{t H}_{3\alpha}(t)(0)}{n_1 a i} - \frac{\widetilde{H}_{3\alpha}(-n_1 a)}{n_1^2 a^2} + \frac{\widetilde{H}_{3\alpha}(0)}{n_1^2 a^2} \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_2=1}^{\infty} e^{-n_2 b \alpha} \left( \frac{\widetilde{t H}_{3\alpha}(t)(0)}{n_2 b i} - \frac{\widetilde{H}_{3\alpha}(-n_2 b)}{n_2^2 b^2} + \frac{\widetilde{H}_{3\alpha}(0)}{n_2^2 b^2} \right) \\
 & + \frac{abc}{32\pi^3} \sum_{n_3=1}^{\infty} e^{-n_3 c \alpha} \left( \frac{\widetilde{t H}_{3\alpha}(t)(0)}{n_3 c i} - \frac{\widetilde{H}_{3\alpha}(-n_3 c)}{n_3^2 c^2} + \frac{\widetilde{H}_{3\alpha}(0)}{n_3^2 c^2} \right) \\
 & + \frac{abc}{128\pi^3} \widetilde{t^2 H}_{3\alpha}(t)(0)
 \end{aligned}$$

By Lemma 2.6 and  $\widetilde{H}(-x) = \widetilde{H}(x)$ , we have

$$I_{111} + I_{333}$$

$$\begin{aligned}
 &= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\
 &\quad \left( \frac{e^{3n_1 a\alpha} \tilde{H}(n_1 a)}{(n_1 a - n_2 b)(n_1 a - n_3 c)} + \frac{e^{3n_2 b\alpha} \tilde{H}(n_2 b)}{(n_2 b - n_1 a)(n_2 b - n_3 c)} + \frac{e^{3n_3 c\alpha} \tilde{H}(n_3 c)}{(n_3 c - n_1 a)(n_3 c - n_2 b)} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( 3\alpha \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1 a} - \sum_{n_1=1}^{\infty} \frac{e^{2n_1 a\alpha} \tilde{H}(n_1 a)}{n_1^2 a^2} + \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1^2 a^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( 3\alpha \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2 b} - \sum_{n_2=1}^{\infty} \frac{e^{2n_2 b\alpha} \tilde{H}(n_2 b)}{n_2^2 b^2} + \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2^2 b^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( 3\alpha \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3 c} - \sum_{n_3=1}^{\infty} \frac{e^{2n_3 c\alpha} \tilde{H}(n_3 c)}{n_3^2 c^2} + \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2 c^2} \right) \\
 &\quad + \frac{abc}{128\pi^3} t^2 \widetilde{H}_{3\alpha}(t)(0).
 \end{aligned} \tag{2.12}$$

Similarly we compute  $I_{113} + I_{331}$ ,  $I_{131} + I_{313}$  and  $I_{311} + I_{133}$ :

$$\begin{aligned}
 &I_{113} + I_{331} \\
 &= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\
 &\quad \left( \frac{e^{-n_1 a\alpha} \tilde{H}(n_1 a)}{(n_1 a + n_2 b)(n_1 a + n_3 c)} + \frac{e^{n_2 b\alpha} \tilde{H}(n_2 b)}{(n_2 b + n_1 a)(n_2 b - n_3 c)} + \frac{e^{n_3 c\alpha} \tilde{H}(n_3 c)}{(n_3 c + n_1 a)(n_3 c - n_2 b)} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1 a} - \sum_{n_1=1}^{\infty} \frac{e^{-2n_1 a\alpha} \tilde{H}(n_1 a)}{n_1^2 a^2} + \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1^2 a^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2 b} - \sum_{n_2=1}^{\infty} \frac{\tilde{H}(n_2 b)}{n_2^2 b^2} + \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2^2 b^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3 c} - \sum_{n_3=1}^{\infty} \frac{\tilde{H}(n_3 c)}{n_3^2 c^2} + \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2 c^2} \right) \\
 &\quad + \frac{abc}{128\pi^3} t^2 \widetilde{H}_{\alpha}(t)(0),
 \end{aligned} \tag{2.13}$$

$$\begin{aligned}
 &I_{131} + I_{313} \\
 &= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\
 &\quad \left( \frac{e^{n_1 a\alpha} \tilde{H}(n_1 a)}{(n_1 a + n_2 b)(n_1 a - n_3 c)} + \frac{e^{-n_2 b\alpha} \tilde{H}(n_2 b)}{(n_2 b + n_1 a)(n_2 b + n_3 c)} + \frac{e^{n_3 c\alpha} \tilde{H}(n_3 c)}{(n_3 c - n_1 a)(n_3 c + n_2 b)} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1 a} - \sum_{n_1=1}^{\infty} \frac{\tilde{H}(n_1 a)}{n_1^2 a^2} + \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a\alpha}}{n_1^2 a^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2 b} - \sum_{n_2=1}^{\infty} \frac{e^{-2n_2 b\alpha} \tilde{H}(n_2 b)}{n_2^2 b^2} + \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b\alpha}}{n_2^2 b^2} \right) \\
 &\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3 c} - \sum_{n_3=1}^{\infty} \frac{\tilde{H}(n_3 c)}{n_3^2 c^2} + \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c\alpha}}{n_3^2 c^2} \right)
 \end{aligned}$$

$$+ \frac{abc}{128\pi^3} \widetilde{t^2 H_\alpha(t)}(0), \quad (2.14)$$

$$\begin{aligned}
& I_{311} + I_{133} \\
&= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b + n_3 c)\alpha} \times \\
&\quad \left( \frac{e^{n_1 a \alpha} \tilde{H}(n_1 a)}{(n_1 a - n_2 b)(n_1 a + n_3 c)} + \frac{e^{n_2 b \alpha} \tilde{H}(n_2 b)}{(n_2 b - n_1 a)(n_2 b + n_3 c)} + \frac{e^{-n_3 c \alpha} \tilde{H}(n_3 c)}{(n_3 c + n_1 a)(n_3 c + n_2 b)} \right) \\
&\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a \alpha}}{n_1 a} - \sum_{n_1=1}^{\infty} \frac{\tilde{H}(n_1 a)}{n_1^2 a^2} + \tilde{H}(0) \sum_{n_1=1}^{\infty} \frac{e^{-n_1 a \alpha}}{n_1^2 a^2} \right) \\
&\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b \alpha}}{n_2 b} - \sum_{n_2=1}^{\infty} \frac{\tilde{H}(n_2 b)}{n_2^2 b^2} + \tilde{H}(0) \sum_{n_2=1}^{\infty} \frac{e^{-n_2 b \alpha}}{n_2^2 b^2} \right) \\
&\quad + \frac{abc}{32\pi^3} \left( \alpha \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c \alpha}}{n_3 c} - \sum_{n_3=1}^{\infty} \frac{e^{-2n_3 c \alpha} \tilde{H}(n_3 c)}{n_3^2 c^2} + \tilde{H}(0) \sum_{n_3=1}^{\infty} \frac{e^{-n_3 c \alpha}}{n_3^2 c^2} \right) \\
&\quad + \frac{abc}{128\pi^3} \widetilde{t^2 H_\alpha(t)}(0).
\end{aligned} \quad (2.15)$$

Considering the limit  $\alpha \rightarrow 0$  in the sum of (2.12), (2.13), (2.14) and (2.15), we have

$$\begin{aligned}
& \lim_{\alpha \rightarrow 0} (I_{111} + I_{113} + I_{131} + I_{133} + I_{311} + I_{313} + I_{331} + I_{333} + \frac{abc}{8\pi^3} (K_1(\alpha) + K_2(\alpha) + K_3(\alpha))) \\
&= -\frac{abc}{8\pi^3} \left( \sum_{n_1=1}^{\infty} \frac{\tilde{H}(n_1 a)}{n_1^2 a^2} + \sum_{n_2=1}^{\infty} \frac{\tilde{H}(n_2 b)}{n_2^2 b^2} + \sum_{n_3=1}^{\infty} \frac{\tilde{H}(n_3 c)}{n_3^2 c^2} \right) \\
&\quad + \frac{abc}{8\pi^3} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \zeta(2) \tilde{H}(0) - \frac{abc}{32\pi^3} \widetilde{t^2 H(t)}(0),
\end{aligned} \quad (2.16)$$

where

$$\begin{aligned}
K_1(\alpha) &= \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_2 b + n_3 c)\alpha} \times \\
&\quad \left( \frac{e^{n_1 a \alpha} (e^{n_1 a \alpha} + e^{-n_1 a \alpha})^2 n_1^2 a^2}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} + \frac{(e^{3n_1 a \alpha} - e^{-n_1 a \alpha}) n_1 n_2 a b}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \right. \\
&\quad \left. + \frac{(e^{3n_1 a \alpha} - e^{-n_1 a \alpha}) n_1 n_3 a c}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} + \frac{e^{n_1 a \alpha} (e^{n_1 a \alpha} - e^{-n_1 a \alpha})^2 n_2 n_3 b c}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \right) \tilde{H}(n_1 a), \\
K_2(\alpha) &= \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_3 c)\alpha} \times \\
&\quad \left( \frac{e^{n_2 b \alpha} (e^{n_2 b \alpha} + e^{-n_2 b \alpha})^2 n_2^2 b^2}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} + \frac{(e^{3n_2 b \alpha} - e^{-n_2 b \alpha}) n_1 n_2 a b}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} \right. \\
&\quad \left. + \frac{(e^{3n_2 b \alpha} - e^{-n_2 b \alpha}) n_2 n_3 b c}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} + \frac{e^{n_2 b \alpha} (e^{n_2 b \alpha} - e^{-n_2 b \alpha})^2 n_1 n_3 a c}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} \right) \tilde{H}(n_2 b), \\
K_3(\alpha) &= \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} e^{-(n_1 a + n_2 b)\alpha} \times
\end{aligned}$$

## EULER PRODUCT EXPRESSION OF TRIPLE ZETA FUNCTIONS

$$\left( \frac{e^{n_3 c \alpha} (e^{n_3 c \alpha} + e^{-n_3 c \alpha})^2 n_3^2 c^2}{(n_3^2 c^2 - n_1^2 a^2)(n_3^2 c^2 - n_2^2 b^2)} + \frac{(e^{3n_3 c \alpha} - e^{-n_3 c \alpha}) n_1 n_3 a c}{(n_3^2 c^2 - n_1^2 a^2)(n_3^2 c^2 - n_2^2 b^2)} \right. \\ \left. + \frac{(e^{3n_3 c \alpha} - e^{-n_3 c \alpha}) n_2 n_3 b c}{(n_3^2 c^2 - n_1^2 a^2)(n_3^2 c^2 - n_2^2 b^2)} + \frac{e^{n_3 c \alpha} (e^{n_3 c \alpha} - e^{-n_3 c \alpha})^2 n_1 n_2 a b}{(n_3^2 c^2 - n_1^2 a^2)(n_3^2 c^2 - n_2^2 b^2)} \right) \tilde{H}(n_3 c),$$

and  $\zeta(s)$  is the Riemann zeta function. Estimating  $|K_j(\alpha) - K_j(0)|$  by using Lemma 2.5 and using  $\tilde{H}(x) \ll \mu^x$  for some  $\mu \in (0,1)$ , we have

$$\lim_{\alpha \rightarrow 0} K_j(\alpha) = K_j(0) \quad (2.17)$$

for  $j = 1, 2, 3$ . By (2.16) and (2.17), we have

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} (I_{111} + I_{113} + I_{131} + I_{133} + I_{311} + I_{313} + I_{331} + I_{333}) \\ &= -\frac{abc}{8\pi^3} \sum_{(n_1, n_2, n_3) \in \Gamma} \varepsilon_{n_1, n_2, n_3} \times \\ & \quad \left( \frac{4n_1^2 a^2}{(n_1^2 a^2 - n_2^2 b^2)(n_1^2 a^2 - n_3^2 c^2)} \tilde{H}(n_1 a) + \frac{4n_2^2 b^2}{(n_2^2 b^2 - n_1^2 a^2)(n_2^2 b^2 - n_3^2 c^2)} \tilde{H}(n_2 b) \right. \\ & \quad \left. + \frac{4n_3^2 c^2}{(n_3^2 c^2 - n_1^2 a^2)(n_3^2 c^2 - n_2^2 b^2)} \tilde{H}(n_3 c) \right) \\ & - \frac{abc}{8\pi^3} \left( \sum_{n_1=1}^{\infty} \frac{\tilde{H}(n_1 a)}{n_1^2 a^2} + \sum_{n_2=1}^{\infty} \frac{\tilde{H}(n_2 b)}{n_2^2 b^2} + \sum_{n_3=1}^{\infty} \frac{\tilde{H}(n_3 c)}{n_3^2 c^2} \right) \\ & + \frac{abc}{48\pi} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \tilde{H}(0) - \frac{abc}{32\pi^3} \widetilde{t^2 H(t)}(0). \end{aligned} \quad (2.18)$$

By

$$\sum_{n>0} \frac{2ku}{k^2 u^2 - n^2 v^2} + \frac{1}{ku} = \frac{\pi}{v} \cot \left( \pi \frac{ku}{v} \right),$$

(2.18) turns to

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} (I_{111} + I_{113} + I_{131} + I_{133} + I_{311} + I_{313} + I_{331} + I_{333}) \\ &= -\frac{a}{8\pi} \sum_{n_1>0} \cot \left( \pi \frac{n_1 a}{b} \right) \cot \left( \pi \frac{n_1 a}{c} \right) \tilde{H}(n_1 a) \\ & \quad - \frac{b}{8\pi} \sum_{n_2>0} \cot \left( \pi \frac{n_2 b}{c} \right) \cot \left( \pi \frac{n_2 b}{a} \right) \tilde{H}(n_2 b) \\ & \quad - \frac{c}{8\pi} \sum_{n_3>0} \cot \left( \pi \frac{n_3 c}{a} \right) \cot \left( \pi \frac{n_3 c}{b} \right) \tilde{H}(n_3 c) \\ & \quad + \frac{abc}{48\pi} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \tilde{H}(0) - \frac{abc}{32\pi^3} \tilde{H}''(0). \end{aligned} \quad (2.19)$$

Next we calculate  $I_{i_1 i_2 i_3}$  with  $i_1 = 2$ ,  $i_2 = 2$  or  $i_3 = 2$ . Put

$$\begin{aligned} I^{(1)} &:= \sum_{j,k=1}^3 I_{2jk}, \quad I^{(2)} := \sum_{j,k=1}^3 I_{j2k}, \quad I^{(3)} := \sum_{j,k=1}^3 I_{jk2}, \\ I^{(4)} &:= \sum_{j=1}^3 I_{22j}, \quad I^{(5)} := \sum_{j=1}^3 I_{2j2}, \quad I^{(6)} := \sum_{j=1}^3 I_{j22}. \end{aligned}$$

Then we have

$$\begin{aligned} I^{(1)} &= \frac{1}{(2\pi i)^3} \int_{C_2} \int_{\partial D} \int_{\partial D} h(s_1 + s_2 + s_3) \frac{Z'_a}{Z_a}(s_1) \frac{Z'_b}{Z_b}(s_2) \frac{Z'_c}{Z_c}(s_3) ds_1 ds_2 ds_3 \\ &= \frac{1}{2\pi i} \int_{C_2} \sum_{\rho_a, \rho_b} h(\rho_a + \rho_b + s_3) \frac{Z'_c}{Z_c}(s_3) ds_3 \\ &= \frac{1}{2\pi i} \int_{\pi}^0 \sum_{\rho_a, \rho_b > 0} h(\rho_a + \rho_b + \alpha e^{i\theta}) \frac{Z'_c}{Z_c}(\alpha e^{i\theta}) \alpha ie^{i\theta} d\theta, \end{aligned}$$

where  $\rho_a, \rho_b$  run through the zeros of  $Z_a(s), Z_b(s)$  with  $\text{Im}(\rho_a) > 0, \text{Im}(\rho_b) > 0$ , respectively. As  $\alpha \rightarrow 0$ , we have

$$\lim_{\alpha \rightarrow 0} I^{(1)} = \frac{1}{2\pi i} \int_{\pi}^0 \sum_{\rho_a, \rho_b} h(\rho_a + \rho_b) id\theta = -\frac{1}{2} \sum_{\rho_a, \rho_b} h(\rho_a + \rho_b). \quad (2.20)$$

We similarly deal with  $I^{(k)}$  ( $k = 2, 3, \dots, 6$ ) and  $I_{222}$  to get

$$\lim_{\alpha \rightarrow 0} I^{(2)} = -\frac{1}{2} \sum_{\rho_a, \rho_c} h(\rho_a + \rho_c), \quad (2.21)$$

$$\lim_{\alpha \rightarrow 0} I^{(3)} = -\frac{1}{2} \sum_{\rho_b, \rho_c} h(\rho_b + \rho_c). \quad (2.22)$$

$$\lim_{\alpha \rightarrow 0} I^{(4)} = \frac{1}{4} \sum_{\rho_a} h(\rho_a), \quad (2.23)$$

$$\lim_{\alpha \rightarrow 0} I^{(5)} = \frac{1}{4} \sum_{\rho_b} h(\rho_b), \quad (2.24)$$

$$\lim_{\alpha \rightarrow 0} I^{(6)} = \frac{1}{4} \sum_{\rho_a} h(\rho_a), \quad (2.25)$$

$$\lim_{\alpha \rightarrow 0} I_{222} = -\frac{1}{8} h(0), \quad (2.26)$$

where  $\rho_k$  runs over the zeros of  $Z_k(s)$  with  $\text{Im}(\rho_k) > 0$ . By (2.20), (2.21), (2.22), (2.23), (2.24), (2.25) and (2.26), we have

$$\begin{aligned} &\lim_{\alpha \rightarrow 0} \sum_{(i_1, i_2, i_3) \in \{1, 2, 3\}^3 \setminus \{1, 3\}^3} I_{i_1 i_2 i_3} \\ &= \lim_{\alpha \rightarrow 0} (I^{(1)} + I^{(2)} + I^{(3)} - (I^{(4)} + I^{(5)} + I^{(6)}) + I_{222}) \\ &= -\frac{1}{2} \left( \sum_{\rho_a, \rho_b} h(\rho_a + \rho_b) + \sum_{\rho_a, \rho_c} h(\rho_a + \rho_c) + \sum_{\rho_b, \rho_c > 0} h(\rho_b + \rho_c) \right) \\ &\quad - \frac{1}{4} \left( \sum_{\rho_a} h(\rho_a) + \sum_{\rho_b} h(\rho_b) + \sum_{\rho_c} h(\rho_c) \right) - \frac{1}{8} h(0). \end{aligned} \quad (2.27)$$

We apply (2.19) and (2.27) to the limit  $\alpha \rightarrow 0$  in (2.5). This completes the proof.  $\square$

### 3. Expression of Triple Sine

We define the multiple Hurwitz zeta function due to Barnes as

$$\zeta_r(s, z, \underline{\omega}) := \sum_{n_1, \dots, n_r \geq 0} (n_1\omega_1 + \dots + n_r\omega_r + z)^{-s}$$

for  $\underline{\omega} = (\omega_1, \dots, \omega_r) \in (\mathbb{R}_{>0})^r$ , the multiple gamma and the multiple sine as

$$\begin{aligned} \Gamma_r(z, \underline{\omega}) &:= \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \underline{\omega})\Big|_{s=0}\right), \\ S_r(z, \underline{\omega}) &:= \Gamma_r(z, \underline{\omega})^{-1}\Gamma_r(\omega_1 + \dots + \omega_r - z, \underline{\omega})^{(-1)^r}. \end{aligned}$$

$S_3(z, \underline{\omega})$  has a following expression ([KuKo] Proposition 2.4):

$$\begin{aligned} S_3(z, \underline{\omega}) &= e^{Q_{\underline{\omega}}(z)} z \prod'_{n_1, n_2, n_3 \geq 0} P_3\left(-\frac{z}{n_1\omega_1 + n_2\omega_2 + n_3\omega_3}\right) \prod_{n_1, n_2, n_3 \geq 1} P_3\left(\frac{z}{n_1\omega_1 + n_2\omega_2 + n_3\omega_3}\right), \quad (3.1) \end{aligned}$$

where  $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \dots + \frac{u^r}{r})$ , and  $Q_{\underline{\omega}}(z) \in \mathbb{C}[z]$  satisfies  $\deg Q_{\underline{\omega}}(z) \leq 3$ .

LEMMA 3.1.

$$\frac{d^3}{dz^3} \log(1 - e^{iaz}) = 2 \sum_{n=-\infty}^{\infty} \frac{1}{(z - \frac{2\pi n}{a})^3}.$$

*Proof.* By

$$\log(1 - e^{iaz}) = -\frac{\pi i}{2} + \frac{az}{2}i + \log\left(2 \sin \frac{az}{2}\right)$$

and

$$2 \sin \frac{az}{2} = az \prod_{n=1}^{\infty} \left(1 - \left(\frac{az}{2\pi n}\right)^2\right)$$

we have

$$\begin{aligned} \frac{d^3}{dz^3} \log(1 - e^{iaz}) &= \frac{2}{z^3} + 2 \sum_{n=1}^{\infty} \left( \frac{1}{(z - \frac{2\pi n}{a})^3} + \frac{1}{(z + \frac{2\pi n}{a})^3} \right) \\ &= 2 \sum_{n=-\infty}^{\infty} \frac{1}{(z - \frac{2\pi n}{a})^3}. \end{aligned}$$

□

LEMMA 3.2. ([KoKu] Theorem 2.3) Let  $I(t)$  be odd function in  $L^1(\mathbb{R})$  with satisfying following (i), (ii):

(i)  $I(t) = O(t^{-2})$  ( $|t| \rightarrow \infty$ )

(ii) There exists  $\mu \in (0, 1)$  such that  $\tilde{I}(x) = O(\mu^x)$  ( $x \rightarrow \infty$ ).

Let  $a, b$  be positive real numbers such that  $a/b$  is generic. Then we have

$$\begin{aligned} &\sum_{k, n > 0} I\left(2\pi\left(\frac{k}{a} + \frac{n}{b}\right)\right) + \frac{1}{2} \left( \sum_{k > 0} I\left(2\pi\frac{k}{a}\right) + \sum_{n > 0} I\left(2\pi\frac{n}{b}\right) \right) \\ &= -\frac{ia}{4\pi} \sum_{k > 0} \cot\left(\pi\frac{ka}{b}\right) \tilde{I}(ka) - \frac{ib}{4\pi} \sum_{n > 0} \cot\left(\pi\frac{nb}{a}\right) \tilde{I}(nb) - \frac{iab}{8\pi^2} \tilde{I}'(0). \quad (3.2) \end{aligned}$$

LEMMA 3.3. ([KoKu] Theorem 3.2) Assume  $\omega_1/\omega_2$  is generic, then the double sine function has the following expression in  $\text{Im}(z) > 0$ :

$$\begin{aligned} S_2(z, (\omega_1, \omega_2)) = & \exp \left( \frac{1}{2i} \sum_{k=1}^{\infty} \frac{1}{k} \cot \left( \pi k \frac{\omega_2}{\omega_1} \right) e^{2\pi i k \frac{z}{\omega_1}} \right. \\ & + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{1}{n} \cot \left( \pi n \frac{\omega_1}{\omega_2} \right) e^{2\pi i n \frac{z}{\omega_2}} \\ & + \frac{1}{2} \log(1 - e^{2\pi i \frac{z}{\omega_1}}) + \frac{1}{2} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) \\ & \left. + \frac{\pi i z^2}{2\omega_1 \omega_2} - \frac{\pi i}{2} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) z + \frac{\pi i}{12} \left( \frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 3 \right) \right). \end{aligned}$$

THEOREM 3.4. Let  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  be in  $(\mathbb{R}_{>0})^3$  such that  $\omega_i/\omega_j$  are generic ( $i, j = 1, 2, 3; i \neq j$ ). Then the triple sine function has the following expression in  $\text{Im}(z) > 0$ :

$$\begin{aligned} S_3(z, \underline{\omega}) = & \exp \left( \frac{1}{4} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot \left( \pi \frac{n_1 \omega_2}{\omega_1} \right) \cot \left( \pi \frac{n_1 \omega_3}{\omega_1} \right) e^{2\pi i n_1 \frac{z}{\omega_1}} \right. \\ & + \frac{1}{4} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot \left( \pi \frac{n_2 \omega_1}{\omega_2} \right) \cot \left( \pi \frac{n_2 \omega_3}{\omega_2} \right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\ & + \frac{1}{4} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot \left( \pi \frac{n_3 \omega_1}{\omega_3} \right) \cot \left( \pi \frac{n_3 \omega_2}{\omega_3} \right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\ & + \frac{1}{4i} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \left( \cot \left( \pi \frac{n_1 \omega_2}{\omega_1} \right) + \cot \left( \pi \frac{n_1 \omega_3}{\omega_1} \right) \right) e^{2\pi i n_1 \frac{z}{\omega_1}} \\ & + \frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \left( \cot \left( \pi \frac{n_2 \omega_1}{\omega_2} \right) + \cot \left( \pi \frac{n_2 \omega_3}{\omega_2} \right) \right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\ & + \frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \left( \cot \left( \pi \frac{n_3 \omega_1}{\omega_3} \right) + \cot \left( \pi \frac{n_3 \omega_2}{\omega_3} \right) \right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\ & + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_1}}) + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) \\ & + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_3}}) - \frac{\pi i z^3}{6\omega_1 \omega_2 \omega_3} + \frac{\pi i}{4} \left( \frac{1}{\omega_1 \omega_2} + \frac{1}{\omega_2 \omega_3} + \frac{1}{\omega_3 \omega_1} \right) z^2 \\ & - \frac{\pi i}{12} \left( \frac{3}{\omega_1} + \frac{3}{\omega_2} + \frac{3}{\omega_3} + \frac{\omega_1}{\omega_2 \omega_3} + \frac{\omega_2}{\omega_3 \omega_1} + \frac{\omega_3}{\omega_1 \omega_2} \right) z \\ & \left. + \frac{\pi i}{24} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} + 3 \right) \right) \end{aligned}$$

*Proof.* By (3.1) we have

$$\begin{aligned} & \frac{d^3}{dz^3} (\log S_3(z, \underline{\omega})) \\ &= C_{\underline{\omega}} + \frac{2}{z^3} \\ &+ 2 \sum_{n_1, n_2, n_3 \geq 1} \left( \frac{1}{(z + (n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3))^3} + \frac{1}{(z - (n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3))^3} \right) \end{aligned}$$

$$\begin{aligned}
 & +2 \left( \sum_{n_1, n_2 \geq 1} \frac{1}{(z + (n_1\omega_1 + n_2\omega_2))^3} + \sum_{n_2, n_3 \geq 1} \frac{1}{(z + (n_2\omega_2 + n_3\omega_3))^3} \right. \\
 & \quad \left. + \sum_{n_1, n_3 \geq 1} \frac{1}{(z + (n_1\omega_1 + n_3\omega_3))^3} \right) \\
 & +2 \left( \sum_{n_1 \geq 1} \frac{1}{(z + n_1\omega_1)^3} + \sum_{n_2 \geq 1} \frac{1}{(z + n_2\omega_2)^2} + \sum_{n_3 \geq 1} \frac{1}{(z + n_3\omega_3)^3} \right), \tag{3.3}
 \end{aligned}$$

where  $C_{\underline{\omega}}$  is a constant depending on  $\underline{\omega}$ .

Put

$$H(t) = \frac{1}{(z+t)^3} + \frac{1}{(z-t)^3} \quad (\operatorname{Im}(z) > 0).$$

As we have

$$\tilde{H}(x) = 2\pi i \operatorname{Res}_{t=z} (H(t)e^{itx}) = \pi ix^2 e^{ixz} \quad (x \geq 0)$$

and  $H(t) = O(t^{-4})(|t| \rightarrow \infty)$ ,  $H(t)$  satisfies (i) and (ii) in §2. As  $\tilde{H}''(0) = 2\pi i$ , by putting

$$\begin{aligned}
 F(z) := & \sum_{n_1, n_2, n_3 \geq 1} \left( \frac{1}{(z + (n_1\omega_1 + n_2\omega_2 + n_3\omega_3))^3} + \frac{1}{(z - (n_1\omega_1 + n_2\omega_2 + n_3\omega_3))^3} \right) \\
 & + \frac{1}{2} \sum_{n_1, n_2 \geq 1} \left( \frac{1}{(z + (n_1\omega_1 + n_2\omega_2))^3} + \frac{1}{(z - (n_1\omega_1 + n_2\omega_2))^3} \right) \\
 & + \frac{1}{2} \sum_{n_2, n_3 \geq 1} \left( \frac{1}{(z + (n_2\omega_2 + n_3\omega_3))^3} + \frac{1}{(z - (n_2\omega_2 + n_3\omega_3))^3} \right) \\
 & + \frac{1}{2} \sum_{n_1, n_3 \geq 1} \left( \frac{1}{(z + (n_1\omega_1 + n_3\omega_3))^3} + \frac{1}{(z - (n_1\omega_1 + n_3\omega_3))^3} \right) \\
 & + \frac{1}{4} \sum_{n_1 \geq 1} \left( \frac{1}{(z + n_1\omega_1)^3} + \frac{1}{(z - n_1\omega_1)^3} \right) \\
 & + \frac{1}{4} \sum_{n_2 \geq 1} \left( \frac{1}{(z + n_2\omega_2)^3} + \frac{1}{(z - n_2\omega_2)^3} \right) \\
 & + \frac{1}{4} \sum_{n_3 \geq 1} \left( \frac{1}{(z + n_3\omega_3)^3} + \frac{1}{(z - n_3\omega_3)^3} \right) + \frac{1}{4z^3},
 \end{aligned}$$

the summation formula (2.3) shows

$$\begin{aligned}
 F(z) = & -\frac{a}{8} \sum_{n_1=1}^{\infty} \cot\left(\pi \frac{n_1 a}{b}\right) \cot\left(\pi \frac{n_1 a}{b}\right) i(n_1 a)^2 e^{in_1 az} \\
 & -\frac{b}{8} \sum_{n_2=1}^{\infty} \cot\left(\pi \frac{n_2 b}{a}\right) \cot\left(\pi \frac{n_2 b}{c}\right) i(n_2 b)^2 e^{in_2 bz} \\
 & -\frac{c}{8} \sum_{n_3=1}^{\infty} \cot\left(\pi \frac{n_3 c}{a}\right) \cot\left(\pi \frac{n_3 c}{b}\right) i(n_3 c)^2 e^{in_3 cz} - \frac{abc}{16\pi^2} i \\
 = & \frac{d^3}{dz^3} \left( \frac{1}{8} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi \frac{n_1 a}{b}\right) \cot\left(\pi \frac{n_1 a}{c}\right) e^{in_1 az} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{8} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 b}{a}\right) \cot\left(\pi \frac{n_2 b}{c}\right) e^{i n_2 b z} \\
 & + \frac{1}{8} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 c}{a}\right) \cot\left(\pi \frac{n_3 c}{b}\right) e^{i n_3 c z} \Big) - \frac{abc}{16\pi^2} i
 \end{aligned}$$

By (3.1) with  $\omega_1 = \frac{2\pi}{a}$ ,  $\omega_2 = \frac{2\pi}{b}$  and  $\omega_3 = \frac{2\pi}{c}$ , we have

$$\begin{aligned}
 & \frac{d^3}{dz^3} (\log S_3(z, \underline{\omega})) \\
 & = C_{\underline{\omega}} + 2F(z) + \frac{3}{2z^3} + \sum_{n_1, n_2 \geq 1} \left( \frac{1}{(z + (n_1 \omega_1 + n_2 \omega_2))^3} - \frac{1}{(z - (n_1 \omega_1 + n_2 \omega_2))^3} \right) \\
 & + \sum_{n_2, n_3 \geq 1} \left( \frac{1}{(z + (n_2 \omega_2 + n_3 \omega_3))^3} - \frac{1}{(z - (n_2 \omega_2 + n_3 \omega_3))^3} \right) \\
 & + \sum_{n_1, n_3 \geq 1} \left( \frac{1}{(z + (n_1 \omega_1 + n_3 \omega_3))^3} - \frac{1}{(z - (n_1 \omega_1 + n_3 \omega_3))^3} \right) \\
 & + \frac{3}{2} \left( \sum_{n_1 \geq 1} \frac{1}{(z + n_1 \omega_1)^3} + \sum_{n_2 \geq 1} \frac{1}{(z + n_2 \omega_2)^3} + \sum_{n_3 \geq 1} \frac{1}{(z + n_3 \omega_3)^3} \right) \\
 & - \frac{1}{2} \left( \sum_{n_1 \geq 1} \frac{1}{(z - n_1 \omega_1)^3} + \sum_{n_2 \geq 1} \frac{1}{(z - n_2 \omega_2)^3} + \sum_{n_3 \geq 1} \frac{1}{(z - n_3 \omega_3)^3} \right). \tag{3.4}
 \end{aligned}$$

Putting

$$I(t) = \frac{1}{(z+t)^3} - \frac{1}{(z-t)^3},$$

$I(t)$  is a odd function in  $L^1(\mathbb{R})$  and satisfies  $I(t) = O(t^{-2})$  ( $|t| \rightarrow \infty$ ). Besides,

$$\tilde{I}(x) = 2\pi i \operatorname{Res}_{t=z} (I(t) e^{itx}) = -\pi i x^2 e^{ixz} \quad (x \geq 0).$$

Hence  $I(t)$  satisfies the assumption of Lemma 3.2. By putting

$$\begin{aligned}
 G_1(z) & := \sum_{n_1, n_2 \geq 1} A(n_1 \omega_1 + n_2 \omega_2) + \frac{1}{2} \left( \sum_{n_1 \geq 1} A(n_1 \omega_1) + \sum_{n_2 \geq 1} A(n_2 \omega_2) \right), \\
 G_2(z) & := \sum_{n_2, n_3 \geq 1} A(n_2 \omega_2 + n_3 \omega_3) + \frac{1}{2} \left( \sum_{n_2 \geq 1} A(n_2 \omega_2) + \sum_{n_3 \geq 1} A(n_3 \omega_3) \right), \\
 G_3(z) & := \sum_{n_1, n_3 \geq 1} A(n_1 \omega_1 + n_3 \omega_3) + \frac{1}{2} \left( \sum_{n_1 \geq 1} A(n_1 \omega_1) + \sum_{n_3 \geq 1} A(n_3 \omega_3) \right),
 \end{aligned}$$

Lemma 3.2 shows

$$\begin{aligned}
 G_1(z) & = \frac{1}{4i} \sum_{n_1=1}^{\infty} \cot\left(\pi \frac{n_1 a}{b}\right) (in_1 a)^3 e^{i n_1 a z} + \frac{1}{4i} \sum_{n_2=1}^{\infty} \cot\left(\pi \frac{n_2 b}{a}\right) (in_2 b)^3 e^{i n_2 b z} \\
 & = \frac{d^3}{dz^3} \left( \frac{1}{4i} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi \frac{n_1 a}{b}\right) e^{i n_1 a z} + \frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 b}{a}\right) e^{i n_2 b z} \right). \tag{3.5}
 \end{aligned}$$

Similarly we compute

$$G_2(z) = \frac{d^3}{dz^3} \left( \frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 b}{c}\right) e^{in_2 bz} + \frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 c}{b}\right) e^{in_3 cz} \right), \quad (3.6)$$

$$G_3(z) = \frac{d^3}{dz^3} \left( \frac{1}{4i} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi \frac{n_1 a}{c}\right) e^{in_1 az} + \frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 c}{a}\right) e^{in_3 cz} \right). \quad (3.7)$$

We express (3.4) using  $G_1, G_2$  and  $G_3$ :

$$\begin{aligned} & \frac{d^3}{dz^3} (\log S_3(z, \underline{\omega})) \\ &= C_{\underline{\omega}} + 2F(z) + G_1(z) + G_2(z) + G_3(z) \\ &+ \frac{1}{2} \left( \sum_{n_1=-\infty}^{\infty} \frac{1}{(z - n_1 \omega_1)^3} + \sum_{n_2=-\infty}^{\infty} \frac{1}{(z - n_2 \omega_2)^3} + \sum_{n_3=-\infty}^{\infty} \frac{1}{(z - n_3 \omega_3)^3} \right). \end{aligned} \quad (3.8)$$

By (3.4), (3.5), (3.6), (3.7) and Lemma 3.1,

$$\begin{aligned} Q(z) := \log S_3(z, \underline{\omega}) - \frac{1}{4} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \cot\left(\pi \frac{n_1 \omega_2}{\omega_1}\right) \cot\left(\pi \frac{n_1 \omega_3}{\omega_1}\right) e^{2\pi i n_1 \frac{z}{\omega_1}} \\ - \frac{1}{4} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) \cot\left(\pi \frac{n_2 \omega_3}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\ - \frac{1}{4} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) \cot\left(\pi \frac{n_3 \omega_2}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\ - \frac{1}{4i} \sum_{n_1=1}^{\infty} \frac{1}{n_1} \left( \cot\left(\pi \frac{n_1 \omega_2}{\omega_1}\right) + \cot\left(\pi \frac{n_1 \omega_3}{\omega_1}\right) \right) e^{2\pi i n_1 \frac{z}{\omega_1}} \\ - \frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \left( \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) + \cot\left(\pi \frac{n_2 \omega_3}{\omega_2}\right) \right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\ - \frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \left( \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) + \cot\left(\pi \frac{n_3 \omega_2}{\omega_3}\right) \right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\ - \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_1}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_3}}) \end{aligned} \quad (3.9)$$

is a polynomial of degree at most three. Thus we put  $Q(z) = a + bz + cz^2 + dz^3$  and will compute  $a, b, c$  and  $d$ . First we compute  $b, c$  and  $d$  by considering

$$Q(z + \omega_1) - Q(z) = (b\omega_1 + c\omega_1^2 + d\omega_1^3) + (2c\omega_1 + 3d\omega_1^2)z + 3d\omega_1 z^2. \quad (3.10)$$

By (3.9) we have

$$\begin{aligned} & Q(z + \omega_1) - Q(z) \\ &= \log \frac{S_3(z + \omega_1, \underline{\omega})}{S_3(z, \underline{\omega})} \\ &- \frac{1}{4} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) \cot\left(\pi \frac{n_2 \omega_3}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} (e^{2\pi i n_2 \frac{\omega_1}{\omega_2}} - 1) \\ &- \frac{1}{4} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) \cot\left(\pi \frac{n_3 \omega_2}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} (e^{2\pi i n_3 \frac{\omega_1}{\omega_3}} - 1) \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \left( \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) + \cot\left(\pi \frac{n_2 \omega_3}{\omega_2}\right) \right) e^{2\pi i n_2 \frac{z}{\omega_2}} (e^{2\pi i n_2 \frac{\omega_1}{\omega_2}} - 1) \\
 & -\frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \left( \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) + \cot\left(\pi \frac{n_3 \omega_2}{\omega_3}\right) \right) e^{2\pi i n_3 \frac{z}{\omega_3}} (e^{2\pi i n_3 \frac{\omega_1}{\omega_3}} - 1) \\
 & -\frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_2}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_3}}) \\
 & + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_3}}). \tag{3.11}
 \end{aligned}$$

Using the formula

$$\cot(x) = i \frac{e^{2\pi i x} + 1}{e^{2\pi i x} - 1}, \tag{3.12}$$

(3.11) turns to

$$\begin{aligned}
 & Q(z + \omega_1) - Q(z) \\
 & = \log \frac{S_3(z + \omega_1, \underline{\omega})}{S_3(z, \underline{\omega})} \\
 & -\frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} (e^{2\pi i n_2 \frac{\omega_1}{\omega_2}} - 1) \\
 & + \frac{1}{2i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 \omega_3}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} \\
 & -\frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} (e^{2\pi i n_3 \frac{\omega_1}{\omega_3}} - 1) \\
 & + \frac{1}{2i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 \omega_2}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} \\
 & -\frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_2}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_3}}) \\
 & + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) + \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_3}}).
 \end{aligned}$$

Using the formula [KuKo, (2.4)] we get

$$S_3(z + \omega_1, \underline{\omega}) S_3(z, \underline{\omega})^{-1} = S_2(z, (\omega_2, \omega_3))^{-1}. \tag{3.13}$$

By Lemma 3.3 and (3.13), we have

$$\begin{aligned}
 & Q(z + \omega_1) - Q(z) \\
 & = -\frac{1}{4i} \sum_{n_2=1}^{\infty} \frac{1}{n_2} \cot\left(\pi \frac{n_2 \omega_1}{\omega_2}\right) e^{2\pi i n_2 \frac{z}{\omega_2}} (e^{2\pi i n_2 \frac{\omega_1}{\omega_2}} - 1) \\
 & -\frac{1}{4i} \sum_{n_3=1}^{\infty} \frac{1}{n_3} \cot\left(\pi \frac{n_3 \omega_1}{\omega_3}\right) e^{2\pi i n_3 \frac{z}{\omega_3}} (e^{2\pi i n_3 \frac{\omega_1}{\omega_3}} - 1) \\
 & -\frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_2}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z+\omega_1}{\omega_3}}) \\
 & -\frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_2}}) - \frac{1}{4} \log(1 - e^{2\pi i \frac{z}{\omega_3}}) \\
 & -\frac{\pi i z^2}{2\omega_2 \omega_3} + \frac{\pi i}{2} \left( \frac{1}{\omega_2} + \frac{1}{\omega_3} \right) z - \frac{\pi i}{12} \left( \frac{\omega_3}{\omega_2} + \frac{\omega_2}{\omega_3} + 3 \right). \tag{3.14}
 \end{aligned}$$

By (3.12) we have

$$Q(z + \omega_1) - Q(z) = -\frac{\pi i z^2}{2\omega_2\omega_3} + \frac{\pi i}{2} \left( \frac{1}{\omega_2} + \frac{1}{\omega_3} \right) z - \frac{\pi i}{12} \left( \frac{\omega_3}{\omega_2} + \frac{\omega_2}{\omega_3} + 3 \right). \quad (3.15)$$

Comparing the coefficients of (3.15), we have

$$\begin{aligned} b &= -\frac{\pi i}{12} \left( \frac{3}{\omega_1} + \frac{3}{\omega_2} + \frac{3}{\omega_3} + \frac{\omega_1}{\omega_2\omega_3} + \frac{\omega_2}{\omega_3\omega_1} + \frac{\omega_3}{\omega_1\omega_2} \right), \\ c &= \frac{\pi i}{4} \left( \frac{1}{\omega_1\omega_2} + \frac{1}{\omega_2\omega_3} + \frac{1}{\omega_3\omega_1} \right), \\ d &= -\frac{\pi i}{6\omega_1\omega_2\omega_3}. \end{aligned}$$

Next we will treat  $a$  by considering

$$\begin{aligned} &Q(z) + Q(z + \frac{\omega_1}{2}) + Q(z + \frac{\omega_2}{2}) + Q(z + \frac{\omega_3}{2}) + Q(z + \frac{\omega_1 + \omega_2}{2}) \\ &+ Q(z + \frac{\omega_2 + \omega_3}{2}) + Q(z + \frac{\omega_3 + \omega_1}{2}) + Q(z + \frac{\omega_1 + \omega_2 + \omega_3}{2}) \\ &- Q(2z). \end{aligned} \quad (3.16)$$

The constant term of (3.16) is

$$7a - \frac{7\pi i}{24\omega_1\omega_2\omega_3} (\omega_1^2\omega_2 + \omega_1\omega_2^2 + \omega_2^2\omega_3 + \omega_2\omega_3^2 + \omega_3^2\omega_1 + \omega_3\omega_1^2 + 3\omega_1\omega_2\omega_3). \quad (3.17)$$

On the other hand we will compute (3.16) by using (3.9). Putting

$$\begin{aligned} A_0 &:= \log \frac{\prod_{k_1, k_2, k_3=0}^1 S_3 \left( z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2}, \underline{\omega} \right)}{S_3(2z, \underline{\omega})}, \\ A_1 &:= -\frac{1}{4} \sum_{n_1=1}^{\infty} \left\{ \frac{1}{n_1} \cot \left( \pi \frac{n_1\omega_2}{\omega_1} \right) \cot \left( \pi \frac{n_1\omega_3}{\omega_1} \right) \right. \\ &\quad \times \left. \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_1(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_1}} - e^{\frac{4\pi i n_1}{\omega_1}} \right) \right\} \\ &\quad - \frac{1}{4i} \sum_{n_1=1}^{\infty} \left\{ \frac{1}{n_1} \left( \cot \left( \pi \frac{n_1\omega_2}{\omega_1} \right) + \cot \left( \pi \frac{n_1\omega_3}{\omega_1} \right) \right) \right. \\ &\quad \times \left. \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_1(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_1}} - e^{\frac{4\pi i n_1}{\omega_1}} \right) \right\}, \\ A_2 &:= -\frac{1}{4} \sum_{n_2=1}^{\infty} \left\{ \frac{1}{n_2} \cot \left( \pi \frac{n_2\omega_1}{\omega_2} \right) \cot \left( \pi \frac{n_2\omega_3}{\omega_2} \right) \right. \\ &\quad \times \left. \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_2(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_2}} - e^{\frac{4\pi i n_2}{\omega_2}} \right) \right\} \\ &\quad - \frac{1}{4i} \sum_{n_2=1}^{\infty} \left\{ \frac{1}{n_2} \left( \cot \left( \pi \frac{n_2\omega_1}{\omega_2} \right) + \cot \left( \pi \frac{n_2\omega_3}{\omega_2} \right) \right) \right. \end{aligned}$$

$$\begin{aligned}
 & \times \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_2(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_2}} - e^{\frac{4\pi i n_2}{\omega_2}} \right) \Bigg\}, \\
 A_3 := & -\frac{1}{4} \sum_{n_3=1}^{\infty} \left\{ \frac{1}{n_3} \cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right) \right. \\
 & \times \left. \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_3(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_3}} - e^{\frac{4\pi i n_3}{\omega_3}} \right) \right\} \\
 & - \frac{1}{4i} \sum_{n_3=1}^{\infty} \left\{ \frac{1}{n_3} \left( \cot\left(\pi \frac{n_3\omega_1}{\omega_3}\right) + \cot\left(\pi \frac{n_3\omega_2}{\omega_3}\right) \right) \right. \\
 & \times \left. \left( \sum_{k_1, k_2, k_3=0}^1 e^{\frac{2\pi i n_3(z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_3}} - e^{\frac{4\pi i n_3}{\omega_3}} \right) \right\}, \\
 A_4 := & -\frac{1}{4} \log \frac{\prod_{k_1, k_2, k_3=0}^1 \left( 1 - e^{\frac{2\pi i (z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_1}} \right)}{1 - e^{\frac{4\pi i z}{\omega_1}}}, \\
 A_5 := & -\frac{1}{4} \log \frac{\prod_{k_1, k_2, k_3=0}^1 \left( 1 - e^{\frac{2\pi i (z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_2}} \right)}{1 - e^{\frac{4\pi i z}{\omega_2}}}, \\
 A_6 := & -\frac{1}{4} \log \frac{\prod_{k_1, k_2, k_3=0}^1 \left( 1 - e^{\frac{2\pi i (z + \frac{k_1\omega_1 + k_2\omega_2 + k_3\omega_3}{2})}{\omega_3}} \right)}{1 - e^{\frac{4\pi i z}{\omega_3}}},
 \end{aligned}$$

we write (3.16) as  $\sum_{j=0}^6 A_j$ . The formula [KuKo, (2.5)] gives  $A_0 = 0$ . Computing  $A_1$  by (3.12), we have

$$\begin{aligned}
 A_1 = & -\frac{1}{4} \sum_{n_1=1}^{\infty} \left\{ \frac{1}{n_1} \left( e^{\frac{4\pi i n_1(z + \frac{\omega_2 + \omega_3}{2})}{\omega_1}} + e^{\frac{4\pi i n_1(z + \frac{\omega_2}{2})}{\omega_1}} + e^{\frac{4\pi i n_1(z + \frac{\omega_3}{2})}{\omega_1}} \right) \right\} \\
 = & \frac{1}{4} \log \left\{ (1 - e^{\frac{4\pi i (z + \frac{\omega_2 + \omega_3}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z + \frac{\omega_2}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z + \frac{\omega_3}{2})}{\omega_1}}) \right\}.
 \end{aligned}$$

$A_4$  is easily computed as

$$A_4 = -\frac{1}{4} \log \left\{ (1 - e^{\frac{4\pi i (z + \frac{\omega_2 + \omega_3}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z + \frac{\omega_2}{2})}{\omega_1}})(1 - e^{\frac{4\pi i (z + \frac{\omega_3}{2})}{\omega_1}}) \right\}.$$

Therefore  $A_1 + A_4 = 0$ . Similarly computing we have  $A_2 + A_5 = A_3 + A_6 = 0$ . Hence  $\sum_{j=0}^6 A_j = 0$ . Therefore its constant term (3.17) vanishes, which lead to

$$a = \frac{\pi i}{24} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} + 3 \right).$$

□

#### 4. Proof of Theorem 1.2

Let  $Z_j (j = 1, \dots, r)$  be meromorphic functions of order  $\mu_j$ . We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod'_{\rho \in \mathbb{C}} P_{\mu_j} \left( \frac{s}{\rho} \right)^{m_j(\rho)},$$

where  $P_r(u) := (1-u) \exp(u + \frac{u^2}{2} + \dots + \frac{u^r}{r})$ ,  $m_j$  denotes the multiplicity function for  $Z_j$ ,  $k_j := m_j(0)$  and  $Q_j$  is a polynomial with  $\deg Q_j(s) \leq \mu_j$ . The product  $\prod'$  means  $\lim_{R \rightarrow \infty} \prod_{0 < |\rho| < R}$ . Then, we have

$$\begin{aligned} Z_1(s) \otimes \dots \otimes Z_r(s) &= s^{k_1 \dots k_r} e^{Q(s)} \\ &\times \prod'_{\rho_1, \dots, \rho_r \in \mathbb{C}} P_{\mu_1 + \dots + \mu_r} \left( \frac{s}{\rho_1 + \dots + \rho_r} \right)^{m(\rho_1, \dots, \rho_r)}, \end{aligned}$$

where  $Q(s)$  is a polynomial with  $\deg Q(s) \leq \mu_1 + \dots + \mu_r$  and

$$m(\rho_1, \dots, \rho_r) := m_1(\rho_1) \dots m_r(\rho_r) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \dots, \operatorname{Im}(\rho_r) \geq 0, \\ (-1)^{r-1} & \text{if } \operatorname{Im}(\rho_1), \dots, \operatorname{Im}(\rho_r) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 4.1. *The absolute tensor product of Hasse zeta function for finite fields is given as follows:*

$$\zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r) = e^{Q(s)} S_3 \left( is, \left( \frac{2\pi}{\log p}, \frac{2\pi}{\log q}, \frac{2\pi}{\log r} \right) \right)^{-1},$$

where  $p, q, r$  are powers of prime and  $Q(s) \in \mathbb{C}[s]$  satisfies  $\deg Q(s) \leq 3$  depending on  $p, q, r$ .

*Proof.* We compute that the Hadamard product for Hasse zeta function is given by

$$\zeta(s, \mathbb{F}_p) = s^{-1} e^{\tilde{Q}_p(s)} \prod'_{n=-\infty}^{\infty} P_1 \left( \frac{s}{\frac{2\pi i}{\log p} n} \right)^{-1},$$

where  $\tilde{Q}_j(s)$  is a linear polynomial depending on  $p$ . Thus by the definition of the absolute tensor product,

$$\begin{aligned} \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r) &= s^{-1} e^{\tilde{Q}_{p,q,r}(s)} \times \\ &\prod'_{n_1, n_2, n_3 \in \mathbb{Z}} P_3 \left( \frac{s}{\frac{2\pi i}{\log p} n_1 + \frac{2\pi i}{\log q} n_2 + \frac{2\pi i}{\log r} n_3} \right)^{m_{n_1, n_2, n_3}}, \end{aligned}$$

where  $\tilde{Q}_{p,q,r}(s) \in \mathbb{C}[s]$  satisfies  $\deg \tilde{Q}_{p,q,r}(s) \leq 3$  and

$$\begin{aligned} m_{n_1, n_2, n_3} &= m \left( \frac{2\pi i}{\log p} n_1, \frac{2\pi i}{\log q} n_2, \frac{2\pi i}{\log r} n_3 \right) \\ &= \begin{cases} -1 & \text{if } n_1, n_2, n_3 \geq 0 \text{ or } n_1, n_2, n_3 < 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \zeta(s, \mathbb{F}_p) \otimes \zeta(s, \mathbb{F}_q) \otimes \zeta(s, \mathbb{F}_r) &= s^{-1} e^{\tilde{Q}_{p,q,r}(s)} \prod'_{n_1, n_2, n_3 \geq 0} P_3 \left( \frac{s}{\frac{2\pi i}{\log p} n_1 + \frac{2\pi i}{\log q} n_2 + \frac{2\pi i}{\log r} n_3} \right)^{-1} \times \end{aligned}$$

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$$\prod_{n_1, n_2, n_3 \geq 1} P_3 \left( -\frac{s}{\frac{2\pi i}{\log p} n_1 + \frac{2\pi i}{\log q} n_2 + \frac{2\pi i}{\log r} n_3} \right)^{-1}.$$

By (3.1) the result follows.  $\square$

*Proof of Theorem 1.2.* Applying Theorem 3.4 with  $z = is$  and  $(\omega_1, \omega_2, \omega_3) = (\frac{2\pi}{\log p}, \frac{2\pi}{\log q}, \frac{2\pi}{\log r})$ , Theorem 1.2 follows.  $\square$

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