**Universality of Hecke $L$-functions in the Grossencharacter-aspect**

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**Abstract:** We consider the Hecke $L$-function $L(s, \lambda^m)$ of the imaginary quadratic field $\mathbb{Q}(i)$ with the $m$-th Grossencharacter $\lambda^m$. We obtain the universality property of $L(s, \lambda^m)$ as both $m$ and $t = \mathrm{Im}(s)$ go to infinity.

**Key words:** Hecke $L$-function, universality of zeta functions, Grossencharacter, imaginary quadratic field,

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# 1 Introduction

Voronin [V] discovered the universality property of the Riemann zeta function in 1975, which is stated as follows:

**Voronin's Theorem** *Let $C$ be a compact subset of the strip $\{s = \sigma + i\tau \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$ with connected complement. Let $f(s)$ be a non-vanishing continuous function on $C$ which is analytic in the interior of $C$. Then for any $\varepsilon > 0$,*

$$\varliminf_{T\to\infty} \frac{\mu(\{t \in [0,T] \mid \sup_{s\in C}|\zeta(s+it) - f(s)| < \varepsilon\})}{T} > 0,$$

*where $\mu$ is the Lebesgue measure on $\mathbb{R}$.*

This result was extended to various zeta functions. The first author proved it for Hecke $L$-functions with ideal class characters [M1] and for those with Grossencharacters [M2]. The universality properties are also generalized to various aspects of zeta functions. Recently Nagoshi proved them for automorphic $L$-functions of $GL(2)$ in the aspect where their weight or level of the cusp forms grows [N1]. Nagoshi also generalized it to Maass cusp forms for $GL(2)$ in the aspect of the Laplace eigenvalues [N2].

In this paper we deal with the Hecke $L$-functions $L(s, \lambda^m)$ of $\mathbb{Q}(i)$ with Grossencharacters $\lambda^m$ ($m \in \mathbb{Z}$), where $\lambda$ is a fixed generator of Grossencharacters. We consider the universality property as both $\tau$ and $m$ grow. More precisely our results are stated as follows:

Let $K = \mathbb{Q}(i)$, and for an ideal $\mathfrak{a} = (\alpha) \in K$, the $m$-th Grossencharacter is given by $\lambda^m(\mathfrak{a}) := (\frac{\alpha}{|\alpha|})^{4m}$ for $m \in \mathbb{Z}$. The Hecke $L$-function is defined by $L(s, \lambda^m) = \sum_{\mathfrak{a}} \lambda^m(\mathfrak{a}) N(\mathfrak{a})^{-s}$ for $\sigma = \mathrm{Re}(s) > 1$.

**Theorem 1.1** *Let $C$ be a compact subset in the strip $\{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$. For any function $f(s)$ which is nonzero and continuous on $C$ and which is holomorphic on $\mathrm{Int}(C)$, and for any $\varepsilon > 0$, we have*

$$\varliminf_{T\to\infty} \frac{1}{T^2}\mu'(\{(t,m) \in [0,T] \times \{0,...,[T]\} \mid \max_{s\in C}|L(s+it, \lambda^m) - f(s)| < \varepsilon\}) > 0, \tag{1.1}$$

*where $\mu'$ is the product measure on $\mathbb{R} \times \mathbb{Z}$.*

**Remark 1.2** *(a) It is possible to extend Theorem 1.1 to any imaginary quadratic field $K$ of class number one, and to general Hecke character $\chi\lambda^m$ with nontrivial narrow class character $\chi$.*

*(b) In case $K$ is a general number field of finite degree,* (1.1) *would be formulated as follows: Let $n = [K : \mathbb{Q}]$ and $\lambda_1, ..., \lambda_{n-1}$ be a fixed set of generators of Grossencharacters of $K$. Put $\lambda^m = \lambda_1^{m_1} \cdots \lambda_{n-1}^{m_{n-1}}$ for $m=(m_1, ..., m_{n-1}) \in \mathbb{Z}^{n-1}$. Then under the above settings we would have*

$$\varliminf_{T\to\infty} \frac{1}{T^n}\mu'(\{(t,m) \in [0,T]^n \mid \max_{s\in C}|L(s+it, \lambda^m) - f(s)| < \varepsilon\}) > 0$$

*with $\mu'$ the product measure on $\mathbb{R} \times \mathbb{Z}^{n-1}$. This will be treated in the forthcoming paper [M3].*

*(c) In Theorem 1.1, it is unfortunate that the range of $m$ and $t$ must be the same. The universality in the $m$-aspect with $t$ being fixed should also be proved. Difficulty lies in the proof of the mean value theorem for Dirichlet series over $O_K$ twisted by $\lambda^m$. Duke it in [D, Theoreom 1.1], where he takes the average over $(m,t) \in$*

$\{0, ..., [T]\} \times [0, T]$. *He conjectures that the mean value theorem should hold in case of* $(m, t) \in \{0, ..., M\} \times [0, T]$. *We see from the proof of our Theorem 1.1 that Duke's conjecture would imply the universality in the* $m$*-aspect.*

(d) *The Grossencharacter-aspect is also considered in a different context. Petridis and Sarnak [PS] obtain a subconvexity estimate of automorphic L-functions* $L(s, \phi)$ *for a Maass cusp form* $\phi$ *of* $SL(2, Z[i])$*. In order to prove it they consider the twists with Grossencharacters and take an avarage* $\sum \int |L(\frac{1}{2} + it, \phi \otimes \lambda^m)|^2 dt$*, where the summation and the integration is taken over certain range of* $(m, t)$*. Consequently they succeed in obtaining subconvexities in the both* $m$ *and* $t$ *aspects.*

## 2 Propositions

For describing the proof of our main result, we put for $z > 0$

$$L_z(s, \lambda^m) := \prod_{N(\mathfrak{p}) \le z} \left(1 - \frac{\lambda^m(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

where $\mathfrak{p}$ denotes a prime ideal. Theorem 1.1 is an immediate consequence of the following propositions:

**Proposition 2.1** *For any* $\varepsilon > 0$ *there exists* $z_0 > 0$ *such that for any* $z > z_0$

$$\max_{s \in C} |\log L(s + it, \lambda^m) - \log L_z(s + it, \lambda^m)| < \varepsilon$$

*holds as* $T \to \infty$ *for any* $(t, m)$ *in a subset of* $[0, T] \times \{0, ..., [T]\}$ *with positive deinsity which is greater than* $1 - \varepsilon$.

**Proposition 2.2** *For any* $\varepsilon > 0$ *there exists* $z_1 > 0$ *such that for any* $z > z_1$

$$\max_{s \in C} |\log L_z(s + it, \lambda^m) - \log f(s)| < \varepsilon$$

*holds as* $T \to \infty$ *for any* $(t, m)$ *in a subset of* $[0, T] \times \{0, ..., [T]\}$ *with positive density which depends only on* $\varepsilon$.

Since the intersection of the sets of $(t, m)$ in Propositions 2.1 and 2.2 has a positive density, Theorem 1.1 follows.

## 3 Proof of Proposition 2.1

Put $a_m(n)$ to be the coefficient in the Dirichlet series expansion of $L(s, \lambda^m)$: $L(s, \lambda^m) = \sum_{n=1}^{\infty} a_m(n) n^{-s}$. We use the following approximate functional equation of Ramachandra type:

**Lemma 3.1** *For* $s = \sigma + it$ *and* $x, y > 0$, $xy = t^2$, *under the conditions that* $\sigma < \alpha < 2$, $0 < \beta < \sigma$, $0 < \gamma < 2$, *we have*

$$L(s, \lambda^m) = A + B + J_1 + J_2 - \frac{W(m)}{2\pi i} \pi^{2s-1} (J_3 + J_4),$$

*where* $|W(m)| = 1$ *and*

$$A = \sum_{n \le x} \frac{a_m(n)}{n^s},$$

$$B = W(m) \pi^{2s-1} \frac{\Gamma(1 - s + 2m)}{\Gamma(s + 2m)} \sum_{n \le y} \frac{\overline{a_m(n)}}{n^{1-s}},$$

$$J_1 = \frac{1}{2\pi i} \int_{(-\gamma)} x^w \frac{\Gamma(1 + \frac{w}{2})}{w} \sum_{n \le x} \frac{a_m(n)}{n^{s+w}} dw,$$

$$J_2 = \sum_{n > x} \frac{a_m(n)}{n^s} e^{-(n/x)^2},$$

$$J_3 = \frac{1}{2\pi i} \int_{(\beta)} (\pi^2 x)^w \frac{\Gamma(1 - s - w + 2m)}{\Gamma(s + w + 2m)} \frac{\Gamma(1 + \frac{w}{2})}{w} \times \sum_{n \le y} \frac{\overline{a_m(n)}}{n^{1-s-w}} dw,$$

$$J_4 = \frac{1}{2\pi i} \int_{(-\alpha)} (\pi^2 x)^w \frac{\Gamma(1 - s - w + 2m)}{\Gamma(s + w + 2m)} \frac{\Gamma(1 + \frac{w}{2})}{w} \times \sum_{n > y} \frac{\overline{a_m(n)}}{n^{1-s-w}} dw.$$

Let $C_1$ be a compact set in $\{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$ such that $C \subset C_1$. We will compute the integral

$$I = \sum_{m=T}^{2T} \int_T^{2T} \iint_{C_1} \left| \frac{L(s + it, \lambda^m)}{L_z(s + it, \lambda^m)} - 1 \right|^2 d\sigma d\tau dt$$

By changing the order of the integration and the sum, it follows that

$$I = \iint_{C_1} \sum_{m=T}^{2T} \int_T^{2T} \left| \frac{L(s + it, \lambda^m)}{L_z(s + it, \lambda^m)} - 1 \right|^2 dt d\sigma d\tau.$$

By Lemma 3.1 we have

(3.1)

$$\sum_{m=T}^{2T}\int_T^{2T}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)}-1\right|^2 dt$$

$$= \sum_{m=T}^{2T}\int_T^{2T} \left|\frac{A+B+J_1+J_2-\frac{W(m)}{2\pi i}\pi^{2s-1}(J_3+J_4)}{L_z(s+it,\lambda^m)}-1\right|^2 dt$$

$$\ll \sum_{m=T}^{2T}\int_T^{2T}\left|\frac{A}{L_z(s+it,\lambda^m)}-1\right|^2 dt + \sum_{m=T}^{2T}\int_T^{2T}\left|\frac{B}{L_z(s+it,\lambda^m)}\right|^2 dt + \cdots + \sum_{m=T}^{2T}\int_T^{2T}\left|\frac{W(m)}{2\pi i}\pi^{2s-1}\frac{J_4}{L_z(s+it,\lambda^m)}\right|^2 dt$$

We will compute each term in (3.1) which we put as $I_A$, $I_B$, $I_{J_1}$,...,$I_{J_4}$. By putting $x=T$ we have for some coefficients $b_m(n)$ with $|b_m(n)|<n^\varepsilon$ such that

$$L_z(s,\lambda^m)^{-1}\sum_{n\le T}\frac{a_m(n)}{n^s} = 1+\sum_{z<n<z^\varepsilon T}\frac{b_m(n)}{n^s}.$$

Thus

$$I_A = \sum_{m=T}^{2T}\int_T^{2T}\left|\frac{A}{L_z(s+it,\lambda^m)}-1\right|^2 dt = \sum_{m=T}^{2T}\int_T^{2T}\left|\sum_{z<n<z^\varepsilon T}\frac{b_m(n)}{n^s}\right|^2 dt. \tag{3.2}$$

By the theorem of Montgomery-Vaughn, (3.2) is estimated by

$$T\left(T\sum_{z<n<z^\varepsilon T}\frac{1}{n^{2\sigma-\varepsilon}}+\sum_{z<n<z^\varepsilon T}\frac{1}{n^{2\sigma-\varepsilon-1}}\right) \ll T^2(z^{1-2\sigma+\varepsilon}+T^{1-2\sigma+\varepsilon}) \tag{3.3}$$

The contribution $I_B$ from the term $B$ to (3.1) is computed by using Stirling's formula as

$$I_B \ll T^{3-2\sigma+\varepsilon}. \tag{3.4}$$

The third term $I_{J_1}$ from $J_1$ is dealt with by our using the Cauchy inequality as

$$I_{J_1} \ll T^{3-2\sigma+\varepsilon}. \tag{3.5}$$

The remaining terms $I_{J_2}$,...,$I_{J_4}$ are similarly estimated. Taking (3.3), (3.4), (3.5) into account we have

$$\sum_{m=T}^{2T}\iint_{C_1}\int_T^{2T}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)}-1\right|^2 dt d\sigma d\tau \ll_{C_1} T^2(z^{1-2\sigma_1+\varepsilon}+T^{1-2\sigma_1+\varepsilon}),$$

where $\sigma_1=\min\{\sigma\in C_1\}$. Since $\sigma_1>\frac{1}{2}$, by taking $z_0$ as $z_0^{1-2\sigma_1+\varepsilon}=\varepsilon^3$, we have

$$\frac{1}{T^2}\sum_{m=T}^{2T}\int_T^{2T}\left(\iint_{C_1}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)}-1\right|^2 d\sigma d\tau\right) dt<\varepsilon^3 \tag{3.6}$$

for $z>z_0$, $T>T_0(z)$. It follows from (3.6) that there exists a subset $A_T$ of $[0,T]\times\{0,\cdots,[T]\}$ with positive density greater than $1-\varepsilon$ such that

$$\iint_{C_1}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)}-1\right|^2 d\sigma d\tau<\varepsilon^2$$

for any $(t,m)\in A_T$. We then have

$$\max_{s\in C}\left|\frac{L(s+it,\lambda^m)}{L_z(s+it,\lambda^m)}-1\right| \ll_{C,C_1}\varepsilon.$$

This means that

$$\max_{s\in C}|\log L(s+it,\lambda^m)-\log L_z(s+it,\lambda^m)| \ll_{C,C_1}\varepsilon$$

for $(t,m)\in A_T$.

**Remark 3.2** *Duke's conjecture [D] would make it possible to deal with the variables $m$ and $t$ separately.*

# 4 Proof of Proposition 2.2

**Lemma 4.1 (Gonek [G])** *Let $C$ be a simply connected compact set of the strip $\frac{1}{2}<\sigma<1$. Let $h(s)$ be a continuous function on $C$ which is regular on* Int$(C)$. *For any $y>0$ there exist $\nu_0=\nu_0(C,h,y)$ and $\theta_p^{(0)}\in[0,1]$ such that*

$$\max_{s\in C}\left|h(s)-\sum_{\substack{y<p\le\nu\\ p\equiv 1 \pmod 4}}\frac{e(\theta_p^{(0)})}{p^s}\right| \ll_C y^{-\frac{1}{2}}$$

*for any $\nu>\nu_0$, where $p$ denotes the prime numbers.*

**Lemma 4.2 ([KV] Theorems 8.1, 8.2)** *Let $a_n \in \mathbb{R}$ $(1 \le n \le N)$ be linearly independent over $\mathbb{Q}$. Then we have*

*(i) If we put*

$$I_A(T) := \{t \in [0,T] \mid (\{a_1 t\}, ..., \{a_N t\}) \in A\}$$

*for any closed Jordan measurable set $A \subset [0,1]^N$ and for $T > 0$, where $\{x\} = x - [x]$, it holds that $\lim_{T\to\infty} \frac{\mu(I_A(T))}{T} = \mu_N(A)$ with $\mu_N$ the Lebesgue measure on $\mathbb{R}^N$.*

*(ii) Let $\Omega$ be a set of continuous functions on $A$. If $\Omega$ is uniformuly bounded and is equicontinuous, it holds uniformly on $f \in \Omega$ that*

$$\lim_{T\to\infty} \frac{1}{T} \int_{I_A(T)} f(\{a_1 t\}, ..., \{a_N t\}) dt = \int \cdots \int_A f(x_1, ..., x_N) dx_1 \cdots dx_N.$$

**Lemma 4.3** *Let $p \equiv 1 \pmod 4$ and $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}$ a prime ideal in $K$. We put $\theta_p$ as $\lambda(\mathfrak{p}) = e^{i\theta_p}$. Then $\{\theta_p\}_{p\equiv 1 \pmod 4}$ is linearly independent over $\mathbb{Q}$.*

*Proof.* Putting $\mathfrak{p} = (a+bi)$ $(a, b \in \mathbb{Z})$, we have $|\alpha| = \sqrt{p}$ and so $\lambda(\mathfrak{p}) = (\frac{a+bi}{\sqrt{p}})^4$. Thus $\cos\theta_p,\ \sin\theta_p \in \mathbb{Z}[\frac{1}{\sqrt{p}}]$. Assume an algebraic dependence as $M\theta_p = m_1\theta_{p_1} + \cdots + m_r\theta_{p_r}$ with $M, m_1, ..., m_r \in \mathbb{Z}$. Then in the equation $\cos(M\theta_p) = \cos(m_1\theta_{p_1} + \cdots + m_r\theta_{p_r})$, the left hand side belongs to $\mathbb{Z}[\frac{1}{\sqrt{p}}]$, whereas the right hand side is in $\mathbb{Z}[\frac{1}{\sqrt{p_1}}, ..., \frac{1}{\sqrt{p_r}}]$. Hence it holds if and only if $\cos(M\theta_p) \in \mathbb{Z}$. Therefore we have $M = 0$.∎

**Proof of Proposition 2.2** We have

$$\log L_z(s, \lambda^m) = \sum_{\substack{p\le z \\ p\equiv 1 \pmod 4}} \sum_{k=1}^{\infty} \frac{2\cos(km\theta_p)}{kp^s} + \sum_{\substack{p\le z \\ p\equiv 3 \pmod 4}} \sum_{k=1}^{\infty} \frac{1}{kp^{2ks}} + \sum_{k=1}^{\infty} \frac{1}{k2^{ks}}$$

We split the sums over $p \le z$ into the ones over $p \le y$ and $y < p \le z$ with $0 < y < z$. We also divide the sum over $1 \le k < \infty$ into $k = 1$, $2 \le k < N$, and $k \ge N$ with $N = [\sigma \log_2 y]$. For partial sums we have the estimates $\sum_{y<p\le z}\sum_{2\le k<N} \frac{2\cos(km\theta_p)}{kp^s} \ll y^{1-2\sigma}$ and $\sum_{p\le y}\sum_{k\ge N} \frac{2\cos(km\theta_p)}{kp^s} \ll y2^{-N\sigma} \ll y^{1-2\sigma}$. Hence

$$\begin{aligned} \log L_z(s+it, \lambda^m) &= \sum_{\substack{y<p\le z \\ p\equiv 1 \pmod 4}} \frac{2\cos(m\theta_p)}{p^s} + l(s+it, y, m) \\ &\quad + O(y^{1-2\sigma}), \end{aligned} \tag{4.1}$$

where

$$l(s, y, m) = \sum_{\substack{p\le y \\ p\equiv 1 \pmod 4}} \sum_{k\le N} \frac{2\cos(km\theta_p)}{kp^{ks}} + \sum_{\substack{p^2\le y \\ p\equiv 3 \pmod 4}} \sum_{k\le N} \frac{1}{kp^{2ks}} + \sum_{k\le N} \frac{1}{k2^{ks}}. \tag{4.2}$$

We fix sufficiently large $y$ which satisfies $y^{1-2\sigma} < \varepsilon$ and $y^{-\frac{1}{2}} < \varepsilon$. Apply Lemma 4.1 for $h(s) = \frac{1}{2}(g(s) - l(s, y, 0))$ and fix $\nu > \nu_0$. Then for any $z > \nu$,

$$|\log L_z(s+it, \lambda^m) - g(s)|$$

$$\le \left| \sum_{\substack{y<p\le\nu \\ p\equiv 1 \pmod 4}} \frac{2\cos(m\theta_p)}{p^{s+it}} - \sum_{y<p\le\nu} \frac{2e(\theta_p^{(0)})}{p^s} \right| \tag{4.3}$$

$$+ |l(s+it, y, m) - l(s, y, 0)| \tag{4.4}$$

$$+ \left| \sum_{\substack{\nu<p\le z \\ p\equiv 1 \pmod 4}} \frac{2\cos(m\theta_p)}{p^{s+it}} \right| + \varepsilon. \tag{4.5}$$

We first deal with (4.3). It is less than

$$\begin{aligned} &\sum_{\substack{y<p\le\nu \\ p\equiv 1 \pmod 4}} \frac{2}{p^\sigma} \left| \frac{\cos(m\theta_p)}{p^{it}} - e(\theta_p^{(0)}) \right| \\ &= \sum_{\substack{y<p\le\nu \\ p\equiv 1 \pmod 4}} \frac{2}{p^\sigma} \left| \cos(m\theta_p)e^{-it\log p} - e(\theta_p^{(0)}) \right| \end{aligned} \tag{4.6}$$

Hence if we take a sufficiently small $\delta > 0$ and put

$$V_T^{(1)} = \{0 \le m \le T \mid \|m\theta_p\| < \delta \quad (y < p \le \nu,\ p \equiv 1 \pmod 4)\},$$

$$U_T^{(1)} = \left\{ t \in [0,T] \mid \left\| t\frac{\log p}{2\pi} - \theta_p^{(0)} \right\| < \delta \quad (y < p \le \nu,\ \ p \equiv 1 \pmod 4) \right\},$$

then for any $(m, t) \in V_T^{(1)} \times U_T^{(1)}$, it holds that (4.6)$<\varepsilon$. By Lemmas 4.2, 4.3, and the linear indepencence over $\mathbb{Q}$ of $\{\log p\}$, we have

$$(4.7) \quad \lim_{T\to\infty} \frac{\mu'(V_T^{(1)} \times U_T^{(1)})}{T^2} = \#(V^{(1)}) \times \mu(U^{(1)})$$

for some $V^{(1)}, U^{(1)} \subset \mathbb{R}^{\pi(\nu)-\pi(y)}$ with $\pi(x)$ the number of primes not greater than $x$.

Next we consider (4.4). It is less than

$$(4.8) \quad \sum_{\substack{p\le y \\ p\equiv 1 \pmod 4}} \sum_{1\le k\le N} \frac{1}{kp^{k\sigma}} \left| \frac{2\cos(km\theta_p)}{p^{ikt}} - 2 \right| + \sum_{\substack{p^2\le y \\ p\equiv 3 \pmod 4}} \sum_{1\le k\le N} \frac{1}{kp^{2k\sigma}} \left| \frac{1}{p^{2ikt}} - 1 \right| + \sum_{1\le k\le N} \frac{1}{2^{k\sigma}} \left| \frac{1}{2^{ikt}} - 1 \right|.$$

Again we take a sufficiently small $\delta' > 0$ and put

$$V_T^{(2)} = \{0 \le m \le T \mid \|m\theta_p\| < \delta' \quad (p \le y,\ p \equiv 1 \pmod 4)\},$$
$$U_T^{(2)} = \left\{ t \in [0,T] \mid \left\| t\tfrac{\log p}{2\pi} \right\| < \delta'(p \le y) \right\}.$$

Then for any $(m,t) \in V_T^{(2)} \times U_T^{(2)}$, it holds that (4.8)$<\varepsilon$.

We put

$$V_T = \{0 \le m \le T \mid \|m\theta_p\| < \delta \ \ (y < p \le \nu,\ \ p \equiv 1(\mathrm{mod} 4)), \ \|m\theta_p\| < \delta' \ \ (p \le y,\ \ p \equiv 1(\mathrm{mod} 4))\}$$

and

$$U_T = \{t \in [0,T] \mid \left\| t\tfrac{\log p}{2\pi} - \theta_p^{(0)} \right\| < \delta \ \ (y < p \le \nu,\ p \equiv 1(\mathrm{mod} 4)), \ \left\| t\tfrac{\log p}{2\pi} \right\| < \delta' \ \ (p \le y)\}.$$

Then (4.3) and (4.4) are bounded by $\varepsilon$ for any $(m,t) \in V_T \times U_T$, and we have

$$\lim_{T\to\infty} \frac{\#V_T}{T} = \mathrm{vol}(V) = (2\delta)^{\frac{\pi(\nu)-\pi(y)}{2}} (2\delta')^{\frac{\pi(y)}{2}}$$
$$\lim_{T\to\infty} \frac{\mu(U_T)}{T} = \mathrm{vol}(U) = (2\delta)^{\pi(\nu)-\pi(y)} (2\delta')^{\pi(y)}$$

where $U$ and $V$ are subsets of $[0,1]^{\pi(\nu)}$. Here we have proved that (4.3) and (4.4) are less than $\varepsilon$ for any $(m,t)$ in a set with positive density.

Lastly we can check that (4.5) is less than $\varepsilon$ for almost all $(m,t) \in U_T \times V_T$. This completes the proof of the theorem.

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