Certain Series Related to the Triple Sine Function

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Running title. Triple Sine Function

Abstract. We compute special values of Dirichlet series whose coefficients are given by the inverse of certain binomial coefficients via the triple sine function.

2000 Mathematics Subject Classification: 11M06

1 Introduction

In the famous proof of the irrationality of $\zeta(3)$ due to Apéry [1], he used the following expression:

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

We refer to van der Poorten [2] and Koecher [3] for explanations and backgrounds. Also Apéry gave a proof of the irrationality of $\zeta(2)$ by using

$$\zeta(2) = 3\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

We can find in [2] and [3] that

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}} = \frac{1}{3}+\frac{2\sqrt{3}\pi}{27},$$
$$\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}} = \frac{\sqrt{3}\pi}{9},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}} = \frac{\pi^4}{3240}.$$

Unfortunately, it remains open to find the expression for

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

The purpose of this paper is to investigate this problem via the triple sine function $\mathcal{S}_3(x)$ studied in the previous papers [4, 5, 6, 7, 8]. We refer to the excellent survey of Manin [9]. Here the triple sine function is defined as

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left(\left(1 - \frac{x^2}{n^2}\right)^{n^2} e^{x^2} \right), \tag{1.1}$$

which is an entire function of order 3. This reminds us of the usual sine function, and actually we define the first sine function $\mathcal{S}_1(x)$ as

$$\mathcal{S}_1(x) = 2\sin(\pi x) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Our result is

Theorem 1

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = 4\pi^2 \log \mathcal{S}_3\left(\frac{1}{6}\right).$$

This is obtained as a special case of

Theorem 2

$$\sum_{n=1}^{\infty} \frac{(2\sin \pi x)^{2n}}{n^3 \binom{2n}{n}} = 4\pi^2 \log \mathcal{S}_3(x)$$

for $-1/2 \leqq x \leqq 1/2$.

From Theorem 1 we moreover show the following identity:

Theorem 3 *It holds that*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = -\frac{4}{3}\zeta(3) + \frac{\sqrt{3}\pi}{3} \left(2L(2,\chi_6) - L(2,\chi_3)\right),$$

where χ_6 *and* χ_3 *are the nontrivial characters modulo* 6 *and* 3, *respectively, and* $L(s,\chi)$ *is the Dirichlet* L*-function.*

2 The Triple Sine Function

In this section we first recall basic properties of the triple sine function (1.1). We find from the definition that

$$\log \mathcal{S}_3(x) = \frac{x^2}{2} + \sum_{n=1}^{\infty} \left(n^2 \log\left(1 - \frac{x^2}{n^2}\right) + x^2 \right)$$

and thus

$$\frac{\mathcal{S}_3'}{\mathcal{S}_3}(x) = \pi x^2 \cot(\pi x), \tag{2.1}$$

where we used the identity

$$\cot(\pi x) = \frac{x}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{x^2 - n^2}.$$

Hence we have

$$\log \mathcal{S}_3(x) = \pi \int_0^x t^2 \cot(\pi t) dt \tag{2.2}$$

as the both sides vanish when $x = 0$. By

$$\cot(\pi t) = i \frac{1 + e^{-2i\pi t}}{1 - e^{-2i\pi t}} = i \left(1 + 2 \sum_{m=1}^{\infty} e^{-2\pi i m t} \right), \quad (\mathrm{Im}(t) < 0)$$

it holds in $\mathrm{Im}(z) < 0$ that

$$\log \mathcal{S}_3(z) = i \int_0^z \pi t^2 \left(1 + 2 \sum_{m=1}^{\infty} e^{-2\pi i m t} \right) dt,$$

where the contour is taken in $\mathrm{Im}(t) < 0$. By integrating by parts, we compute

$$\int_0^z t^2 e^{\alpha t} dt = \frac{z^2 e^{\alpha z}}{\alpha} - \frac{2z e^{\alpha z}}{\alpha^2} + \frac{2(e^{\alpha z} - 1)}{\alpha^3}.$$

Therefore the following expression holds for $\mathrm{Im}(z) < 0$.

$$\log \mathcal{S}_3(z) = -\frac{2}{(2\pi i)^2} \sum_{n=1}^{\infty} \left(\frac{e^{-2\pi i z n} - 1}{n^3} + 2\pi i z \frac{e^{-2\pi i z n}}{n^2} + \frac{(2\pi i z)^2}{2} \frac{e^{-2\pi i z n}}{n} \right) + \frac{\pi i}{3} z^3.$$

By taking the real part and by continuity, we have for $x \in \mathbb{R}$ $(0 < x < 1)$

$$\log \mathcal{S}_3(x) = \frac{2}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\frac{\cos(2\pi n x) - 1}{n^3} + \frac{2\pi x \sin(2\pi n x)}{n^2} - \frac{(2\pi x)^2 \cos(2\pi n x)}{2n} \right).$$

We appeal to the formula

$$\log(2\sin\pi x) = -\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n}$$

to get

$$\log \mathcal{S}_3(x) = x^2\log(2\sin\pi x) + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi n x)}{n^3} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{n^2} - \frac{1}{2\pi^2}\zeta(3). \qquad (2.3)$$

(We can prove this formula also by showing that both sides are 0 at $x = 1$ and that differentiations of both sides are equal to $\pi x^2\cot(\pi x)$.)

Now, letting $x = \frac{1}{6}$ we obtain,

$$\log \mathcal{S}_3\left(\frac{1}{6}\right) = \frac{1}{2\pi^2}\left(\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{n^3} - \zeta(3)\right) + \frac{1}{6\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2}.$$

The Dirichlet series with coefficients $\sin\frac{n\pi}{3}$ and $\cos\frac{n\pi}{3}$ are calculated as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2} &= \frac{\sqrt{3}}{2}\left(\sum_{n\equiv 1,2\,(\mathrm{mod}\ 6)}\frac{1}{n^2} - \sum_{n\equiv 4,5\,(\mathrm{mod}\ 6)}\frac{1}{n^2}\right)\\
&= \frac{\sqrt{3}}{2}\left(2\left(\sum_{n\equiv 1\,(\mathrm{mod}\ 6)}\frac{1}{n^2} - \sum_{n\equiv 5\,(\mathrm{mod}\ 6)}\frac{1}{n^2}\right) - \left(\sum_{n\equiv 1,4\,(\mathrm{mod}\ 6)}\frac{1}{n^2} - \sum_{n\equiv 2,5\,(\mathrm{mod}\ 6)}\frac{1}{n^2}\right)\right)\\
&= \frac{\sqrt{3}}{2}\left(2L(2,\chi_6) - L(2,\chi_3)\right),
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{n^3} &= \frac{1}{2}\left(\sum_{n\equiv 1,5\,(\mathrm{mod}\ 6)}\frac{1}{n^3} - \sum_{n\equiv 2,4\,(\mathrm{mod}\ 6)}\frac{1}{n^3}\right) + \sum_{n\equiv 0\,(\mathrm{mod}\ 6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\ 6)}\frac{1}{n^3}\\
&= \sum_{n\equiv 1,5\,(\mathrm{mod}\ 6)}\frac{1}{n^3} - \frac{1}{2}\left(\sum_{n\equiv 1,2,4,5\,(\mathrm{mod}\ 6)}\frac{1}{n^3}\right) + \sum_{n\equiv 0\,(\mathrm{mod}\ 6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\ 6)}\frac{1}{n^3}\\
&= L(3,\mathbf{1}_6) - \frac{1}{2}L(3,\mathbf{1}_3) + \sum_{n\equiv 0\,(\mathrm{mod}\ 6)}\frac{1}{n^3} - \sum_{n\equiv 3\,(\mathrm{mod}\ 6)}\frac{1}{n^3}\\
&= (1-2^{-3})(1-3^{-3})\zeta(3) - \frac{1}{2}(1-3^{-3})\zeta(3) + 6^{-3}\zeta(3) - 3^{-3}(1-2^{-3})\zeta(3)\\
&= \frac{1}{3}\zeta(3),
\end{aligned}$$

where $\mathbf{1}_m$ denotes the trivial Dirichlet character modulo m. Thus we have

Theorem 4

$$\log \mathcal{S}_3\left(\frac{1}{6}\right) = \frac{1}{2\pi^2}\left(-\frac{2}{3}\zeta(3) + \frac{\sqrt{3}\pi}{6}\left(2L(2,\chi_6) - L(2,\chi_3)\right)\right).$$

Remark 5 When $x = \frac{1}{2}$ in (2.3), we have

$$\begin{aligned}\log \mathcal{S}_3\left(\frac{1}{2}\right) &= \frac{1}{4}\log 2 + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^3} \\ &= \frac{1}{4}\log 2 - \frac{1}{\pi^2}\sum_{n:odd}\frac{1}{n^3} \\ &= \frac{1}{4}\log 2 - \frac{7}{8\pi^2}\zeta(3).\end{aligned}$$

Hence

$$\zeta(3) = \frac{8\pi^2}{7}\log\left(\mathcal{S}_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}}\right)$$

as in [4], [7], [8].

3 Proofs of Theorems

Theorem 1 is a special case of Theorem 2. Theorem 3 is obtained from Theorems 1 and 4. Hence it suffices to prove Theorem 2. It is known by Euler ([2, p.203], [3, p.62]) that

$$\sum_{n=1}^{\infty}\frac{(2\sin(\pi x))^{2n}}{n^2\binom{2n}{n}} = 2\pi^2 x^2$$

for $-1/2 \leqq x \leqq 1/2$. Therefore it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{2n}{n}^{-1} 2^{2n}(\sin(\pi x))^{2n-1}\cos(\pi x) = 2\pi^2 x^2\cot(\pi x).$$

Integrating both sides, we have

$$\begin{aligned}\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\binom{2n}{n}^{-1} 2^{2n-1}(\sin(\pi x))^{2n} &= 2\pi^2\int_0^x t^2\cot(\pi t)dt \\ &= 2\pi\log\mathcal{S}_3(x)\end{aligned}$$

by the equation (2.2). This completes the proof of Theorem 2.

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