Multiple Zeta Functions I

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Running title. Multiple Zeta Functions
Abstract. We compute the absolute tensor product of the Hasse zeta functions for finite fields.

1 Introduction

Let

$$Z_j(s) = \prod_{\rho \in \mathbb{C}} (s-\rho)^{m_j(\rho)}$$

be "zeta functions" expressed as regularized product, where

$$m_j \; : \; \mathbb{C} \to \mathbb{Z}$$

denotes the multiplicity function for $j = 1, ..., r$. (Later we will specify "zeta functions" to be treated.) As in the previous paper [K2] we define the absolute tensor product $(Z_1 \otimes \cdots \otimes Z_r)(s)$ as

$$(Z_1 \otimes \cdots \otimes Z_r)(s) = \prod_{\rho_1, \cdots, \rho_r \in \mathbb{C}} (s - (\rho_1 + \cdots + \rho_r))^{m(\rho_1, \cdots, \rho_r)}$$

with

$$m(\rho_1, \cdots, \rho_r) = m_1(\rho_1) \cdots m_r(\rho_r) \times \begin{cases} 1 & \mathrm{Im}(\rho_j) \geq 0, \quad (j = 1, ..., r) \\ (-1)^{r-1} & \mathrm{Im}(\rho_j) < 0 \quad (j = 1, ..., r) \\ 0 & \text{otherwise.} \end{cases}$$

We refer to the excellent survey of Manin [M]. We are especially interested in the case of Hasse zeta functions $Z_j(s) = \zeta(s, A_j)$ for rings $A_1, ..., A_r$. We recall that the Hasse zeta function $\zeta(s, A)$ of a ring A is defined to be

$$\zeta(s, A) = \prod_{\mathbf{m}} (1 - N(\mathbf{m})^{-s})^{-1}$$

where $\mathbf{m}$ runs over maximal left ideals of A up to the following equivalence:

$$\mathbf{m}_1 \sim \mathbf{m}_2 \Longleftrightarrow A/\mathbf{m}_1 \text{ and } A/\mathbf{m}_2 \text{ are isomorhpic as left } A\text{-modules,}$$

and $N(\mathbf{m}) = \#\mathrm{End}_{A-\mathrm{mod}}(A/\mathbf{m})$. See [K3] and [F]. (For a commutative ring A, the above $\zeta(s, A)$ coincides with the usual Hasse zeta function

$$\zeta(s, A) = \prod_{\mathbf{m}} (1 - N(\mathbf{m})^{-s})^{-1},$$

when $\mathbf{m}$ runs over maximal ideals of A and $N(\mathbf{m}) = \#(A/\mathbf{m})$.)

For simplicity we write

$$\zeta(s, A_1 \otimes \cdots \otimes A_r) = \zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r).$$

Actually, as was explained by Manin [M], we expect that our multiple zeta function would be the zeta function of the "absolute tensor product"

$$A_1 \otimes_{\mathbf{F}_1} \cdots \otimes_{\mathbf{F}_1} A_r$$

that is the tensor product over the (virtual) "one element field" $\mathbf{F}_1$. In any way, we notice that $\zeta(s, A_1 \otimes \cdots \otimes A_r)$ has the following additive structure on zeros and poles: if $\zeta(s, A_j) = 0$ (resp. ∞) and $\mathrm{Im}(s_j)$ $(j = 1, ..., r)$ have the same signature, then $\zeta(s_1 + \cdots + s_r, A_1 \otimes \cdots \otimes A_r) = 0$ (resp. ∞).

Such an additive structure was crucial in the study of Hasse zeta functions of positive characteristic (congruence zeta functions) pursued by Grothendieck [G] and Deligne [D].

In this Part I, we investigate

$$\zeta(s, \mathbf{F}_p \otimes \mathbf{F}_q) = \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$$

for primes p and q. We prove that it has a kind of Euler product expression in terms of the polylogarithm:

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

Theorem 1.1 *The following expressions hold in* $\mathrm{Re}(s) > 0$ *with some polynomial* $Q(s)$ *of degree at most two:*

(1) When $p \neq q$*, we have*

$$\begin{aligned} \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) &= e^{Q_{p,q}(s)} \left(1 - p^{-s}\right)^{\frac{1}{2}} \left(1 - q^{-s}\right)^{\frac{1}{2}} \\ &\times \exp\left(\frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{\log p}{\log q}\right)}{k} p^{-ks} + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns} \right). \end{aligned}$$

(2) When $p = q$*, we have*

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p) = e^{Q_{p,p}(s)} \left(1 - p^{-s}\right)^{1 - \frac{is \log p}{2\pi}} \exp\left(-\frac{\mathrm{Li}_2(p^{-s})}{2\pi i} \right).$$

In Part II we study

$$\zeta(s, \mathbb{Z} \otimes \mathbb{Z}) = \zeta(s, \mathbb{Z}) \otimes \zeta(s, \mathbb{Z})$$

where $\zeta(s, \mathbb{Z}) = \zeta(s)$ is the Riemann zeta function.

2 Double Poisson Summation Formula with Signature

Let $H(t)$ be an odd function in $L^1(\mathbb{R})$ with $H(t) = O(t^{-2})$ as $|t| \to \infty$. We put

$$\widetilde{H}(u) = \int_{-\infty}^{\infty} H(t) e^{itu} dt.$$

Definition A real number α is called *generic* if and only if

$$\lim_{m\to\infty} \|m\alpha\|^{\frac{1}{m}} = 1,$$

where we put $\|x\| := \min\{|x-n| \ : \ n \in \mathbb{Z}\}$ for $x \in \mathbb{R}$.

Examples 2.1 (1) If $\alpha \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q}$, then α is generic.

(2) Let $x, y \in \overline{\mathbb{Q}} \cap \mathbb{R}$. If $\alpha = \frac{\log x}{\log y} \notin \mathbb{Q}$, then α is generic (Baker [B, Theorem 3.1]).

Lemma 2.2 *Assume α is generic, then the power series*

$$\sum_{n=1}^{\infty} \cot(\pi n \alpha) x^n \tag{2.1}$$

absolutely converges in $|x| < 1$.

Proof. As α is generic, we have $\|n\alpha\|^{-1} = O(e^{\varepsilon n})$ as $n \to \infty$ for any $\varepsilon > 0$. Since $\cot(\pi x) \sim 1/(\pi x)$ as $x \to 0$, we have $\cot(\pi n \alpha) = O(e^{\varepsilon n})$ for any $\varepsilon > 0$.∎

Theorem 2.3 *Assume a/b is generic and that the test function $H(t)$ satisfies*

$$\widetilde{H}(x) = O(\mu^x) \tag{2.2}$$

as $x \to \infty$ for some $0 < \mu < 1$, then we have

$$\begin{aligned} &\sum_{k,n>0} H\left(2\pi\left(\frac{k}{a} + \frac{n}{b}\right)\right) + \frac{1}{2}\left(\sum_{k>0} H\left(2\pi \frac{k}{a}\right) + \sum_{n>0} H\left(2\pi \frac{n}{b}\right)\right) \\ &= -\frac{ia}{4\pi} \sum_{k>0} \cot\left(\pi \frac{ka}{b}\right) \widetilde{H}(ka) - \frac{ib}{4\pi} \sum_{n>0} \cot\left(\pi \frac{nb}{a}\right) \widetilde{H}(nb) - \frac{iab}{8\pi^2} \widetilde{H}'(0). \end{aligned} \tag{2.3}$$

Proof. Put $Z_a(s) = \sinh\left(\frac{as}{2}\right)$ and $Z_b(s) = \sinh\left(\frac{bs}{2}\right)$. Let D_T be the region defined by

$$D_T = \{s \in \mathbb{C} \mid |s| > \alpha,\ |\mathrm{Re}(s)| < \alpha,\ 0 < \mathrm{Im}(s) < T\}$$

with $0 < \alpha < \min\left\{\frac{2\pi}{a}, \frac{2\pi}{b}\right\}$. By Cauchy's theorem we have for an odd function h which is regular in D_T

$$\sum_{0<\mathrm{Im}(\rho_a),\mathrm{Im}(\rho_b)<T} h(\rho_a + \rho_b) = \frac{1}{(2\pi i)^2} \int_{\partial D_T} \int_{\partial D_T} h(s_1 + s_2) \frac{Z_a'}{Z_a}(s_1) \frac{Z_b'}{Z_b}(s_2) ds_1 ds_2, \tag{2.4}$$

where ρ_a and ρ_b denote the zeros of $Z_a(s)$ and $Z_b(s)$, respectively, and the integrals along ∂D_T are taken counter clockwise. Considering the limits as $T \to \infty$ in the both sides of (2.4), we have

$$\sum_{\mathrm{Im}(\rho_a),\mathrm{Im}(\rho_b)>0} h(\rho_a + \rho_b) = \frac{1}{(2\pi i)^2} \int_{\partial D} \int_{\partial D} h(s_1 + s_2) \frac{Z_a'}{Z_a}(s_1) \frac{Z_b'}{Z_b}(s_2) ds_1 ds_2, \tag{2.5}$$

where

$$D = \{s \in \mathbb{C} \mid |\mathrm{Re}(s)| < \alpha,\ |s| > \alpha,\ \mathrm{Im}(s) > 0\}.$$

We decompose $\partial D = C_1 \cup C_2 \cup C_3$ with

$$\begin{aligned} C_1 &= \{s \in \partial D \mid \mathrm{Re}(s) = -\alpha\}, \\ C_2 &= \{s \in \partial D \mid |s| = \alpha\}, \\ C_3 &= \{s \in \partial D \mid \mathrm{Re}(s) = \alpha\}. \end{aligned}$$

We compute each double integral $I_{ij} = \frac{1}{(2\pi i)^2} \int_{C_i} \int_{C_j}$ in (2.5).

First we treat the integral along the vertical lines.

$$I_{33} = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty h(2\alpha + i(t_1 + t_2)) \frac{Z_a'}{Z_a}(\alpha + it_1) \frac{Z_b'}{Z_b}(\alpha + it_2) dt_1 dt_2. \tag{2.6}$$

Since

$$\frac{Z_a'}{Z_a}(\alpha + it_1) = \frac{a}{2} + a \sum_{k=1}^\infty e^{-ka(\alpha + it_1)}$$

and

$$\frac{Z_b'}{Z_b}(\alpha + it_2) = \frac{b}{2} + b \sum_{n=1}^\infty e^{-nb(\alpha + it_2)},$$

by putting $H_\alpha(t) = h(2\alpha + i(t_1 + t_2))$ with $t = t_1 + t_2$, (2.6) turns to

$$I_{33} = \frac{1}{4\pi^2} \sum_{k,n \geq 0} \varepsilon_{k,n} ab \int_0^\infty \int_0^t H_\alpha(t) e^{-ka(\alpha + it_1)} e^{-nb(\alpha + i(t - t_1))} dt_1 dt,$$

where we put

$$\varepsilon_{k,n} = \begin{cases} 1/4 & (k=n=0) \\ 1/2 & (k=0, n\neq 0 \text{ or } k\neq 0, n=0) \\ 1 & (\text{otherwise}) \end{cases} .$$

Thus

$$\begin{aligned} I_{33} &= \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_0^\infty \frac{H_\alpha(t)(e^{-ikat}-e^{-inbt})}{-i(ka-nb)} dt \\ &+ \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_0^\infty tH_\alpha(t) e^{-ikat} dt. \end{aligned} \tag{2.7}$$

We similarly compute that

$$\begin{aligned} I_{11} &= -\frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_0^\infty \frac{H_\alpha(-t)(e^{ikat}-e^{inbt})}{i(ka-nb)} dt \\ &- \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_0^\infty tH_\alpha(-t) e^{ikat} dt. \end{aligned} \tag{2.8}$$

By (2.7) and (2.8) we have

$$\begin{aligned} I_{11}+I_{33} &= \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_{-\infty}^\infty \frac{H_\alpha(t)(e^{-ikat}-e^{-inbt})}{-i(ka-nb)} dt \\ &+ \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_{-\infty}^\infty tH_\alpha(t) e^{-ikat} dt. \\ &= \frac{iab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka\neq nb}} \varepsilon_{k,n} \frac{e^{-(ka+nb)\alpha}}{ka-nb} \left(\widetilde{H_\alpha}(-ka) - \widetilde{H_\alpha}(-nb) \right) \\ &+ \frac{ab}{4\pi^2} \sum_{\substack{k,n\geq 0 \\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \widetilde{tH_\alpha(t)}(-ka). \end{aligned}$$

The assumption that a/b is generic implies that the second sum consists only of the term $k=n=0$.

Next we calculate I_{13}. Since $h(i(t_1+t_2)) = H_0(t_1+t_2)$ and $\dfrac{Z_a'}{Z_a}$ is an odd function, we have

$$\begin{aligned} I_{13} &= \frac{-1}{(2\pi i)^2}\int_{\infty}^{0}\int_{0}^{\infty} h(i(t_1+t_2))\frac{Z_a'}{Z_a}(-\alpha+it_1)\frac{Z_b'}{Z_b}(\alpha+it_2)dt_1dt_2 \\ &= \frac{ab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}e^{-(ka+nb)\alpha}\int_0^\infty \frac{H_0(t)(e^{ikat}-e^{-inbt})}{i(ka+nb)}dt + \frac{ab}{16\pi^2}\int_0^\infty tH_0(t)dt. \end{aligned} \tag{2.9}$$

Similarly

$$I_{31} = \frac{ab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}e^{-(ka+nb)\alpha}\int_0^\infty \frac{H_0(-t)(e^{-ikat}-e^{inbt})}{i(ka+nb)}dt + \frac{ab}{16\pi^2}\int_0^\infty tH_0(t)dt. \tag{2.10}$$

Therefore (2.9) and (2.10) lead to

$$I_{13}+I_{31} = -\frac{iab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}\frac{e^{-(ka+nb)\alpha}}{(ka+nb)}\left(\widetilde{H_0}(ka)-\widetilde{H_0}(-nb)\right) - \frac{iab}{16\pi^2}\widetilde{H_0}'(0).$$

Letting $\alpha\to 0$ gives

$$\lim_{\alpha\to 0}(I_{11}+I_{33}+I_{13}+I_{31}) = -\frac{iab}{2\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}\frac{\widetilde{H_0}(ka)ka-\widetilde{H_0}(nb)nb}{k^2a^2-n^2b^2} - \frac{iab}{8\pi^2}\widetilde{H_0}'(0), \tag{2.11}$$

since $\widetilde{H_0}' = i\widetilde{tH_0(t)}$.

Next we treat $I_2 := I_{21}+I_{22}+I_{23}$. We compute

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{\partial D} h(s_1+s_2)\frac{Z_a'}{Z_a}(s_1)ds_1\right)\frac{Z_b'}{Z_b}(s_2)ds_2 \\ &= \frac{1}{2\pi i}\int_{C_2}\sum_{\rho_a} h(\rho_a+s_2)\frac{Z_b'}{Z_b}(s_2)ds_2, \end{aligned}$$

where ρ_a runs through the zeros of $Z_a(s)$ with $\mathrm{Im}(\rho)>0$. Putting $s_2=\alpha e^{i\theta}$, we reach

$$\lim_{\alpha\to 0} I_2 = \frac{1}{2\pi}\int_\pi^0\sum_{\rho_a} h(\rho_a)d\theta = -\frac{1}{2}\sum_{\rho_a} h(\rho_a). \tag{2.12}$$

We similarly deal with $I_2' := I_{12} + I_{22} + I_{32}$ to get

$$\lim_{\alpha\to 0} I_2' = -\frac{1}{2}\sum_{\rho_b} h(\rho_b). \tag{2.13}$$

The integral I_{22}, which appears in both (2.12) and (2.13), tends to 0 as $\alpha \to 0$. Thus taking (2.11), (2.12) and (2.13) into account, (2.5) equals

$$-\frac{iab}{2\pi^2}\sum_{(k,n)\neq(0,0)} \varepsilon_{k,n}\frac{\widetilde{H_0}(ka)ka - \widetilde{H_0}(nb)nb}{k^2a^2 - n^2b^2} - \frac{iab}{8\pi^2}\widetilde{H_0}'(0) - \frac{1}{2}\sum_{k>0} H_0\left(2\pi\frac{k}{a}\right) - \frac{1}{2}\sum_{n>0} H_0\left(2\pi\frac{n}{b}\right).$$

Theorem follows from the formulas

$$\sum_{n>0}\frac{2ka}{k^2a^2 - n^2b^2} + \frac{1}{ka} = \frac{\pi}{b}\cot\left(\pi\frac{ka}{b}\right),$$

$$\sum_{k>0}\frac{2nb}{n^2b^2 - k^2a^2} + \frac{1}{nb} = \frac{\pi}{a}\cot\left(\pi\frac{nb}{a}\right). \blacksquare$$

3 Expression of Double Sine

We use the multiple Hurwitz zeta function due to Barnes

$$\zeta_r(s, z, \underline{\omega}) = \sum_{n_1,\ldots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s}$$

for $\underline{\omega} = (\omega_1, ..., \omega_r)$ and the definitions of the multiple gamma and the multiple sine:

$$\Gamma_r(z, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \underline{\omega})\bigg|_{s=0}\right),$$

$$S_r(z, \underline{\omega}) = \Gamma_r(z, \underline{\omega})^{-1}\Gamma_r(\omega_1 + \cdots + \omega_r - z, \underline{\omega})^{(-1)^r}.$$

When $r = 2$, we have $\underline{\omega} = (\omega_1, \omega_2)$ and

$$S_2(z, \omega_1, \omega_2) = \Gamma_2(z, \omega_1, \omega_2)^{-1}\Gamma_2(\omega_1 + \omega_2 - z, \omega_1, \omega_2).$$

The double gamma function has an expression

$$\Gamma_2(z, \omega_1, \omega_2)^{-1} = e^{Q_1(z)} z \prod_{n_1,n_2\geq 0}{}' P_2\left(-\frac{z}{n_1\omega_1 + n_2\omega_2}\right)$$

and

$$\Gamma_2(\omega_1+\omega_2-z,\omega_1,\omega_2)^{-1} = e^{Q_2(z)} \prod_{n_1,n_2\geq 1} P_2\left(\frac{z}{n_1\omega_1+n_2\omega_2}\right)$$

where $Q_1(z)$ and $Q_2(z)$ are polynomials of degree 2 and $P_2(u) := (1-u)\exp(u+\frac{u^2}{2})$. We then have

$$S_2(z,\omega_1,\omega_2) = e^{c_0+c_1z+c_2z^2} \frac{z \prod'_{n_1,n_2\geq 0} P_2\left(-\frac{z}{n_1\omega_1+n_2\omega_2}\right)}{\prod_{n_1,n_2\geq 1} P_2\left(\frac{z}{n_1\omega_1+n_2\omega_2}\right)} \tag{3.1}$$

where we put $Q_1(z)-Q_2(z) = c_0+c_1z+c_2z^2$.

Lemma 3.1

$$\frac{d^2}{dz^2}\log(1-e^{iaz}) = -\sum_{n=-\infty}^{\infty} \frac{1}{\left(z-\frac{2\pi n}{a}\right)^2}$$

Proof. Since

$$\log(1-e^{iaz}) = -\frac{\pi i}{2} + \log\left(2\sin\frac{az}{2}\right)$$

and

$$2\sin\frac{az}{2} = az\prod_{n=1}^{\infty}\left(1-\left(\frac{az}{2\pi n}\right)^2\right),$$

we have

$$\frac{d^2}{dz^2}\log(1-e^{iaz}) = -\frac{1}{z^2} - \sum_{n=1}^{\infty}\left(\frac{1}{\left(z-\frac{2\pi n}{a}\right)^2} + \frac{1}{\left(z+\frac{2\pi n}{a}\right)^2}\right) = -\sum_{n=-\infty}^{\infty}\frac{1}{\left(z-\frac{2\pi n}{a}\right)^2}. \blacksquare$$

Theorem 3.2 *Assume ω_1/ω_2 is generic, then the double sine function has the following expression in* $\mathrm{Im}(z)>0$*:*

$$\begin{aligned} S_2(z,(\omega_1,\omega_2)) &= \exp\left(\frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi i k\frac{z}{\omega_1}} + \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi i n\frac{z}{\omega_2}}\right. \\ &+ \frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_1}}\right) + \frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_2}}\right) \\ &+ \left.\frac{\pi i z^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right)\right) \end{aligned}$$

Proof. Apply the odd function

$$H(t) = \frac{1}{(z-t)^2} - \frac{1}{(z+t)^2}$$

with $z \in \mathbb{C}$, $\mathrm{Im}(z) > 0$ to our summation formula (2.3). As we have

$$\widetilde{H}(x) = \int_{-\infty}^{\infty} H(t) e^{ixt} dt = 2\pi i \mathrm{Res}_{t=z} \left(H(t) e^{ixt} \right) = -2\pi x e^{ixz},$$

the condition (2.2) is satisfied. As $\widetilde{H}'(0) = -2\pi$, by putting

$$\begin{aligned} F(z) &= \sum_{k,n\geq 1} \left(\frac{1}{\left(z - 2\pi\left(\frac{k}{a} + \frac{n}{b}\right)\right)^2} - \frac{1}{\left(z + 2\pi\left(\frac{k}{a} + \frac{n}{b}\right)\right)^2} \right) \\ &+ \frac{1}{2} \sum_{k>0} \left(\frac{1}{\left(z - 2\pi\frac{k}{a}\right)^2} - \frac{1}{\left(z + 2\pi\frac{k}{a}\right)^2} \right) + \frac{1}{2} \sum_{n>0} \left(\frac{1}{\left(z - 2\pi\frac{n}{b}\right)^2} - \frac{1}{\left(z + 2\pi\frac{n}{b}\right)^2} \right), \end{aligned}$$

the summation formula (2.3) shows

$$\begin{aligned} F(z) &= \frac{i}{2} \sum_{k>0} \cot\left(\pi \frac{ka}{b}\right) ka^2 e^{ikaz} + \frac{i}{2} \sum_{n>0} \cot\left(\pi \frac{nb}{a}\right) nb^2 e^{inbz} + \frac{iab}{4\pi} \\ &= \frac{d^2}{dz^2} \left(\frac{1}{2i} \sum_{k>0} \frac{1}{k} \cot\left(\pi \frac{ka}{b}\right) e^{ikaz} + \frac{1}{2i} \sum_{n>0} \frac{1}{n} \cot\left(\pi \frac{nb}{a}\right) e^{inbz} \right) + \frac{iab}{4\pi} \end{aligned} \tag{3.2}$$

By (3.1) with $n_1 = k$, $n_2 = n$, $\omega_1 = \frac{2\pi}{a}$ and $\omega_2 = \frac{2\pi}{b}$, we have

$$\begin{aligned} \frac{d^2}{dz^2} \log S_2(z, \omega_1, \omega_2) &= -\frac{1}{z^2} - \sum_{n_1,n_2\geq 1} \left(\frac{1}{(z + n_1\omega_1 + n_2\omega_2)^2} - \frac{1}{(z - (n_1\omega_1 + n_2\omega_2))^2} \right) \\ &- \sum_{n_1\geq 1} \frac{1}{(z + n_1\omega_1)^2} - \sum_{n_2\geq 1} \frac{1}{(z + n_2\omega_2)^2} + 2c_2 \\ &= F(z) - \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{1}{(z - 2\pi\frac{k}{a})^2} - \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2\pi\frac{n}{b})^2} + 2c_2 \\ &= \frac{d^2}{dz^2} \left(\frac{1}{2i} \sum_{k>0} \frac{1}{k} \cot\left(\pi \frac{ka}{b}\right) e^{ikaz} + \frac{1}{2i} \sum_{n>0} \frac{1}{n} \cot\left(\pi \frac{nb}{a}\right) e^{inbz} \right. \\ &+ \left. \frac{1}{2} \log(1 - e^{iaz}) + \frac{1}{2} \log(1 - e^{ibz}) \right) + 2c_2 + \frac{iab}{4\pi}, \end{aligned} \tag{3.3}$$

where we used (3.2) and Lemma 3.1. So if we put

$$\begin{aligned} E(z) &:= \log S_2(z, \omega_1, \omega_2) - \left(\frac{1}{2i} \sum_{k>0} \frac{1}{k} \cot\left(\pi \frac{k\omega_2}{\omega_1}\right) e^{\frac{2\pi i k z}{\omega_1}} + \frac{1}{2i} \sum_{n>0} \frac{1}{n} \cot\left(\pi \frac{n\omega_1}{\omega_2}\right) e^{\frac{2\pi i n z}{\omega_2}} \right. \\ &+ \left. \frac{1}{2} \log\left(1 - e^{\frac{2\pi i}{\omega_1} z}\right) + \frac{1}{2} \log\left(1 - e^{\frac{2\pi i}{\omega_2} z}\right) \right), \end{aligned} \tag{3.4}$$

it holds $\frac{d^2}{dz^2}E(z)$ is constant and that $E(z)$ is a polynomial of degree 2.

Thus we put $E(z) = \alpha + \beta z + \gamma z^2$ and will compute α, β and γ. We first calculate β and γ by considering

$$E(z+\omega_1) - E(z) = (\beta\omega_1 + \gamma\omega_1^2) + 2\gamma\omega_1 z. \tag{3.5}$$

It follows from (3.4) that (3.5) equals

$$\log \frac{S_2(z+\omega_1,\omega_1,\omega_2)}{S_2(z,\omega_1,\omega_2)} - \frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)\left(e^{\frac{2\pi i n\omega_1}{\omega_2}} - 1\right)e^{\frac{2\pi i n z}{\omega_2}}$$
$$-\frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega_2}(z+\omega_1)}\right) + \frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega_2}z}\right).$$

The sum over n is computed as

$$\begin{aligned} -\frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)\left(e^{\frac{2\pi i n\omega_1}{\omega_2}} - 1\right)e^{\frac{2\pi i n z}{\omega_2}} &= -\frac{1}{2}\sum_{n>0}\frac{1}{n}\left(1 + e^{\frac{2\pi i n\omega_1}{\omega_2}}\right)e^{\frac{2\pi i n z}{\omega_2}} \\ &= \frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega_2}(z+\omega_1)}\right) + \frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega_2}z}\right). \end{aligned}$$

We appeal to the formula [KK, (2.4)] to get

$$\frac{S_2(z+\omega_1,\omega_1,\omega_2)}{S_2(z,\omega_1,\omega_2)} = S_1(z,\omega_2)^{-1} = \left(2\sin\frac{\pi z}{\omega_2}\right)^{-1}.$$

Hence (3.5) is equal to

$$\begin{aligned} -\log\left(2\sin\frac{\pi z}{\omega_2}\right) + \log\left(1 - e^{\frac{2\pi i}{\omega_2}z}\right) &= -\log\left(2\sin\frac{\pi z}{\omega_2}\right) + \log\left(-2ie^{\frac{\pi i}{\omega_2}z}\sin\frac{\pi z}{\omega_2}\right) \\ &= -\frac{\pi i}{2} + \frac{\pi i}{\omega_2}z. \end{aligned}$$

Therefore we have

$$\beta\omega_1 + \gamma\omega_1^2 = -\frac{\pi i}{2}$$

and

$$2\gamma\omega_1 = \frac{\pi i}{\omega_2}.$$

We thus obtain

$$\beta = -\frac{\pi i}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)$$

and

$$\gamma = \frac{\pi i}{2\omega_1\omega_2}.$$

Next we deal with α by considering

$$E(z)+E\left(z+\frac{\omega_1}{2}\right)+E\left(z+\frac{\omega_2}{2}\right)+E\left(z+\frac{\omega_1+\omega_2}{2}\right)-E(2z). \tag{3.6}$$

The constant term of (3.6) is

$$3\alpha+\beta(\omega_1+\omega_2)+\gamma\left(\frac{\omega_1}{4}+\frac{\omega_2}{4}+\frac{(\omega_1+\omega_2)^2}{4}\right)=3\alpha-\frac{\pi i}{4}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right). \tag{3.7}$$

On the other hand we will compute (3.6) by using (3.4). We write (3.6) as $\sum_{j=0}^{4} A_j$, where

$$\begin{aligned}
A_0 &= \log\frac{S_2(z)S_2\left(z+\frac{\omega_1}{2}\right)S_2\left(z+\frac{\omega_2}{2}\right)S_2\left(z+\frac{\omega_1+\omega_2}{2}\right)}{S_2(2z)},\\
A_1 &= -\frac{1}{2i}\sum_{k>0}\frac{\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)}{k}\left(e^{\frac{2\pi ikz}{\omega_1}}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_1}{2})}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_2}{2})}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_1+\omega_2}{2})}-e^{\frac{4\pi ikz}{\omega_1}}\right),\\
A_2 &= -\frac{1}{2i}\sum_{n>0}\frac{\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)}{n}\left(e^{\frac{2\pi inz}{\omega_2}}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_2}{2})}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_1}{2})}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_2+\omega_1}{2})}-e^{\frac{4\pi inz}{\omega_2}}\right),\\
A_3 &= -\frac{1}{2}\log\frac{\left(1-e^{\frac{2\pi i}{\omega_1}z}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_1}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_1+\omega_2}{2})}\right)}{1-e^{\frac{4\pi i}{\omega_1}z}},\\
A_4 &= -\frac{1}{2}\log\frac{\left(1-e^{\frac{2\pi i}{\omega_2}z}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_2}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_1+\omega_2}{2})}\right)}{1-e^{\frac{4\pi i}{\omega_2}z}}.
\end{aligned}$$

The formula [KK, (2.5)] gives $A_0=0$. Next A_1 is computed as follows:

$$\begin{aligned}
A_1 &= -\frac{1}{2i}\sum_{\substack{k>0\\ \text{even}}}\frac{1}{k}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\left(2e^{\frac{2\pi ikz}{\omega_1}}+2e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_2}{2})}\right)+\frac{1}{2i}\sum_{k>0}\frac{1}{k}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)e^{\frac{4\pi ikz}{\omega_1}}\\
&= -\frac{1}{2i}\sum_{k>0}\frac{1}{k}\left(\cot\left(\pi\frac{2k\omega_2}{\omega_1}\right)\left(1+e^{\frac{2\pi ik\omega_2}{\omega_1}}\right)-\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\right)e^{\frac{4\pi ikz}{\omega_1}}\\
&= -\frac{1}{2}\sum_{k>0}\frac{1}{k}e^{\frac{2\pi ik\omega_2}{\omega_1}}e^{\frac{4\pi ikz}{\omega_1}}\\
&= \frac{1}{2}\log\left(1-e^{\frac{4\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right),
\end{aligned}$$

where we used an identity

$$\cot 2\theta(1+e^{2i\theta})-\cot\theta=ie^{2i\theta}$$

with $\theta = \pi \dfrac{k\omega_2}{\omega_1}$. Similarly A_2 is calculated as

$$A_2 = \frac{1}{2} \log \left(1 - e^{\frac{4\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right).$$

The remaining terms are easily computed as

$$A_3 = -\frac{1}{2} \log \left(1 - e^{\frac{4\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right),$$

$$A_4 = -\frac{1}{2} \log \left(1 - e^{\frac{4\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right).$$

Hence we deduced that (3.6)$= \sum_{j=0}^{4} A_j = 0$. Therefore its constant term (3.7) vanishes, which leads to

$$\alpha = \frac{\pi i}{12} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + 3 \right). \blacksquare$$

4 Proof of Theorem 1.1

We first describe a more precise definition of the absolute tensor product of meromorphic functions. Let Z_j $(j = 1, 2)$ be meromorphic functions of order μ_j. We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod_{\rho \in \mathbb{C}}{}' P_{\mu_j} \left(\frac{s}{\rho}\right)^{m_j(\rho)}, \tag{4.1}$$

where $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$, m_j denotes the multiplicity function with $k_j := m_j(0)$, and Q_j is a polynomial with $\deg Q_j \le \mu_j$. Here the product over $\rho \in \mathbb{C}$ means $\lim_{R\to\infty} \prod_{0<|\rho|<R} P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}$. The absolute tensor product is defined by

$$(Z_1 \otimes Z_2)(s) := s^{k_1 k_2} e^{Q(s)} \prod_{\rho_1, \rho_2 \in \mathbb{C}}{}' P_{\mu_1+\mu_2} \left(\frac{s}{\rho_1 + \rho_2}\right)^{m(\rho_1,\rho_2)}, \tag{4.2}$$

where $Q(s)$ is a polynomial with $\deg Q \le \mu_1 + \mu_2$ and

$$m(\rho_1, \rho_2) := m_1(\rho_1) m_2(\rho_2) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) \ge 0, \\ -1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we do not give the precise definition of the polynomial $Q(s)$, since it is not necessary for our purpose.

In this section we will compute this absolute tensor product for the Hasse zeta functions for finite fields:

$$\begin{aligned} Z_1(s) &= \zeta(s, \mathbf{F}_p) = (1-p^{-s})^{-1}, \\ Z_2(s) &= \zeta(s, \mathbf{F}_q) = (1-q^{-s})^{-1}, \end{aligned}$$

with p, q primes.

Proposition 4.1 *The absolute tensor product of the Hasse zeta functions for finite fields is given as follows:*

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) = e^{Q(s)} S_2 \left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q} \right) \right),$$

where $Q(s)$ is a polynomial of degree at most two, which depends on p and q.

Proof. We easily compute that the Hadamard product (4.1) for the Hasse zeta function is given by

$$\zeta(s, \mathbf{F}_p) = s^{-1} e^{\tilde{Q}_p(s)} \prod_{n=-\infty}^{\infty}{}' P_1 \left(\frac{s}{\frac{2\pi i}{\log p} n} \right)^{-1}$$

with $\tilde{Q}_p(s)$ a linear polynomial depending on p. Thus by the definition (4.2) of the absolute tensor product,

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) = s e^{\tilde{Q}_{p,q}(s)} \prod_{k,n \in \mathbb{Z}}{}' P_2 \left(\frac{s}{\frac{2\pi i}{\log p} k + \frac{2\pi i}{\log q} n} \right)^{m_{k,n}},$$

where $\tilde{Q}_{p,q}(s)$ is a polynomial of degree at most two and

$$m_{k,n} := m \left(\tfrac{2\pi i}{\log p} k, \tfrac{2\pi i}{\log q} n \right) = \begin{cases} 1 & \text{if } k, n \geq 0 \\ -1 & \text{if } k, n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) = s e^{\tilde{Q}_{p,q}(s)} \frac{\displaystyle\prod_{k,n=0}^{\infty}{}' P_2 \left(\frac{s}{\frac{2\pi i}{\log p} k + \frac{2\pi i}{\log q} n} \right)}{\displaystyle\prod_{k,n=1}^{\infty} P_2 \left(-\frac{s}{\frac{2\pi i}{\log p} k + \frac{2\pi i}{\log q} n} \right)}.$$

We appeal to the $r = 2$ case of the formula [KK, Proposition 2.4]:

$$S_2(z,(\omega_1,\omega_2)) = e^{Q_{\underline{\omega}}(z)} z \frac{\displaystyle\prod_{k,n=0}^{\infty}{}' P_2\left(-\frac{z}{\omega_1 k + \omega_2 n}\right)}{\displaystyle\prod_{k,n=1}^{\infty} P_2\left(\frac{z}{\omega_1 k + \omega_2 n}\right)}$$

where $Q_{\underline{\omega}}(z)$ a polynomial with $\deg Q_{\underline{\omega}} \leq 2$. Putting $z = is$ and $\underline{\omega} = (\omega_1, \omega_2) = (\frac{2\pi}{\log p}, \frac{2\pi}{\log q})$, we reach the proposition.∎

Proof of Theorem 1.1. We take $(\omega_1,\omega_2) = (\frac{2\pi}{\log p}, \frac{2\pi}{\log q})$ in Theorem 3.2. Example 2.1 (2) tells that ω_1/ω_2 is generic, if $p \neq q$. Thus Proposition 4.1 gives the assertion (1). For proving (2), we put $p = q$ in Proposition 4.1. We recall the formulas of the double sine function:

$$\begin{aligned} S_2(z,(\omega,\omega)) &= S_2\left(\frac{z}{\omega},(1,1)\right) && \text{([KK, Theorem 2.1(c)])} \\ &= \mathcal{S}_2\left(\frac{z}{\omega}\right)^{-1} \mathcal{S}_1\left(\frac{z}{\omega}\right), && \text{([KK, Example 3.6])} \end{aligned} \tag{4.3}$$

where $\mathcal{S}_r(z)$ $(r = 1, 2)$ are the primitive multiple sine functions [KK]. We have by definition

$$\mathcal{S}_1(z) = 2\sin \pi z$$

and the expression [KK, Theorem 2.8 (2.12)]:

$$\mathcal{S}_2(z) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z}) + z\log(1 - e^{2\pi i z}) - \frac{\pi i}{2}z^2 - \frac{\zeta(2)}{2\pi i}\right)$$

for $\mathrm{Im}(z) > 0$. Thus putting $z = is$ and $\omega = \frac{2\pi}{\log p}$ in (4.3), we have the conclusion.∎

Remark 4.2 The assertion (2) can also be proved in the same manner as in §2 and §3.

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