Kummer's Formula for Multiple Gamma Functions *

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Running title. Kummer's Formula

Abstract. We generalize Kummer's formula on an expression of the gamma function to that of multiple gamma functions using the functional equation for multiple Hurwitz zeta functions.

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1 Introduction

Kummer's formula for the usual gamma function $\Gamma(x)$ is the following identity:

$$\log\Gamma(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\log n)\sin(2\pi nx)}{n} + \frac{\log(2\pi)+\gamma}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n} + \frac{1}{2}\log(2\pi) \quad (1.1)$$

for $0 < x < 1$, which was discovered by Kummer [9] (1847). A spectacular application is given by letting x be a rational number: we have formulas for special values of Dirichlet L-functions using the gamma function. For example, when $x = 1/4$ we get

$$L'(1,\chi_{-4}) = -\pi\log\Gamma\left(\frac{1}{4}\right) + \frac{\pi}{4}(\gamma + 3\log\pi + 2\log 2)$$

due to Malmstén [20] (1849), where

$$L(s,\chi_{-4}) = \sum_{n:\text{odd}}(-1)^{\frac{n-1}{2}}n^{-s}$$

is the Dirichlet L-function for the non-trivial character χ_{-4} modulo 4. This result is equivalently written as

$$L'(0,\chi_{-4}) = 2\log\Gamma\left(\frac{1}{4}\right) - \frac{3}{2}\log 2 - \log\pi$$

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via the functional equation. We remark that $L(0,\chi_{-4}) = 1/2$ and $L(1,\chi_{-4}) = \pi/4$, so Malmstén's result gives the (difficult) second term of the Taylor expansion for $L(s,\chi_{-4})$ at $s=0$ and $s=1$.

Marmstén's study motivated the following excellent formula

$$\log\Gamma(x) = \zeta'(0,x) + \frac{1}{2}\log(2\pi) \tag{1.2}$$

of Lerch [18] (1894), where

$$\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$$

is the Hurwitz zeta function ([8]) and the differentiation is in the first variable s. Combining this with Kronecker's limit formula Lerch [19] (1897) obtained the formula

$$\sum_{\lambda=1}^{\Delta}\left(\frac{D}{\lambda}\right)\log\Gamma\left(\frac{\lambda}{\Delta}\right) = h\log\Delta - \frac{h}{3}\log 2\pi - \sum_{(a,b,c)}\log a + \frac{2}{3}\sum_{(a,b,c)}\log(\vartheta_1'(0|\alpha)\vartheta_1'(0|\beta)), \tag{1.3}$$

where ϑ_1 is the usual notation of theta function (so ϑ_1' is essentially η^3), $\mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field of discriminant D and class number h, (a,b,c) runs over h quadratic forms corresponding to the ideal classes, and $\Delta = |D|$. (It would be convenient to look at the formula (163) of Landau [17].) In recent years this formula is called "Chowla-Selberg formula" after the 70 years late paper of Chowla-Selberg [2] (1967) with no mention to Lerch [19] unfortunately. Concerning the history around Kummer's formula (1.1) and the above two Lerch's formulae (1.2), (1.3) we refer to the excellent survey [17] written by Landau, which is the famous first paper on the prime ideal theorem generalizing the usual prime number theorem to algebraic number fields.

The purpose of this paper is to present a generalization of Kummer's formula in the case of the multiple gamma function $\Gamma_r(x)$. This multiple gamma function originates from Barnes [1] and it is defined as

$$\Gamma_r(x) = \exp(\zeta_r'(0,x))$$

where

$$\begin{aligned}\zeta_r(s,x) &= \sum_{n_1,\ldots,n_r=0}^{\infty}(n_1+\cdots+n_r+x)^{-s}\\ &= \sum_{n=0}^{\infty}\binom{n+r-1}{r-1}(n+x)^{-s}\end{aligned}$$

is the multiple Hurwitz zeta function. We have many applications of $\Gamma_r(x)$ in Sarnak [23], Voros [25], Shintani [24] and our previous papers [10] - [15]. From $\Gamma_1(x) = \Gamma(x)/\sqrt{2\pi}$, we can write Kummer's formula as follows:

$$\begin{aligned} \log\Gamma_1(x) &= \zeta'(0,x) \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\log n)\sin(2\pi nx)}{n} \\ &\quad+\frac{\log(2\pi)+\gamma}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}. \end{aligned} \tag{1.4}$$

Our result is a generalization of this formula:

Theorem 1 *Let* $0 < x < 1$. *Then we have*

(1)

$$\log\Gamma_2(x) = -\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(\log n)\cos(2\pi nx)}{n^2} - \frac{\log(2\pi)+\gamma-1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^2} + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} + (1-x)\log\Gamma_1(x).$$

(2)

$$\log\Gamma_3(x) = -\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{(\log n)\sin(2\pi nx)}{n^3} - \frac{2\log(2\pi)+2\gamma-3}{8\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^3} - \frac{1}{8\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} + \left(\frac{3}{2}-x\right)\log\Gamma_2(x) - \frac{(x-1)^2}{2}\log\Gamma_1(x).$$

In the text we show some applications. We notice that our method is quite simple to obtain $\zeta_r'(0,x)$ using the functional equation $\zeta_r(s,x) = \xi_r(r-s,x)$ where $\xi_r(s,x)$ is a Dirichlet series intimately related to $\zeta_r(s,x)$:

$$\log\Gamma_r(x) = -\xi_r'(r,x).$$

This natural method was discovered by Hardy [4] to reprove the original Kummer's formula. In [5, 6], Hardy studied the case of the double gamma function, but he did not obtain the generalization of Kummer's formula.

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2 Multiple Gamma Functions

We prepare on the multiple gamma function $\Gamma_r(x)$. We refer to Barnes[1] for the general theory. First we know that the function $\Gamma_r(x)$ is a meromorphic function of order r ($\Gamma_r(x)^{-1}$ being an entire function of order r), and it has a simple pole at $x = 0$:

$$\Gamma_r(x) \sim \frac{1}{\rho_r x} \qquad \text{(as } x \to 0).$$

We remark that Barnes [1] used $\Gamma_r^B(x)$ normalized as

$$\Gamma_r^B(x) = \rho_r \Gamma_r(x) \sim \frac{1}{x} \qquad \text{(as } x \to 0).$$

For later use we calculate ρ_r for $r = 1, 2, 3$ here.

Theorem 2 *(1)* $\rho_1 = \sqrt{2\pi}$.

(2)

$$\rho_2 = \sqrt{2\pi} e^{-\zeta'(-1)} = \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{7\log(2\pi) + \gamma - 1}{12}\right).$$

(3)

$$\rho_3 = \sqrt{2\pi} e^{-\frac{3}{2}\zeta'(-1) - \frac{1}{2}\zeta'(-2)} = \exp\left(\frac{\zeta(3)}{8\pi^2} - \frac{3\zeta'(2)}{4\pi^2} + \frac{5\log(2\pi) + \gamma - 1}{8}\right).$$

Proof. From

$$\zeta_r(s, x+1) = \zeta_r(s, x) - \zeta_{r-1}(s, x)$$

we have

$$\Gamma_r(x+1) = \Gamma_r(x)\Gamma_{r-1}(x)^{-1}.$$

Hence letting $x \to 0$ we get

$$\Gamma_r(1) = \frac{\rho_{r-1}}{\rho_r}.$$

(Here we understand that $\zeta_0(s, x) = x^{-s}$, $\Gamma_0(x) = x^{-1}$ and $\rho_0 = 1$.) The first result $\rho_1 = \sqrt{2\pi}$ follows from Lerch's formula

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}} \sim \frac{1}{\sqrt{2\pi} x} \qquad \text{(as } x \to 0).$$

Now we calculate

$$\rho_2 = \rho_1 \Gamma_2(1)^{-1} = \sqrt{2\pi}\Gamma_2(1)^{-1}$$

and

$$\rho_3 = \rho_2 \Gamma_3(1)^{-1} = \sqrt{2\pi}\Gamma_2(1)^{-1}\Gamma_3(1)^{-1}.$$

Since

$$\begin{aligned}\zeta_2(s,x) &= \sum_{n=0}^{\infty}(n+1)(n+x)^{-s} \\ &= \sum_{n=0}^{\infty}((n+x)+(1-x))(n+x)^{-s} \\ &= \sum_{n=0}^{\infty}(n+x)^{-(s-1)}+(1-x)\sum_{n=0}^{\infty}(n+x)^{-s} \\ &= \zeta(s-1,x)+(1-x)\zeta(s,x)\end{aligned}$$

and

$$\begin{aligned}\zeta_3(s,x) &= \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}(n+x)^{-s} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}((n+x)^2+(3-2x)(n+x)+(x-1)(x-2))(n+x)^{-s} \\ &= \frac{1}{2}\zeta(s-2,x)+\frac{3-2x}{2}\zeta(s-1,x)+\frac{(x-1)(x-2)}{2}\zeta(s,x),\end{aligned}$$

we obtain

$$\zeta_2'(0,x)=\zeta'(-1,x)+(1-x)\zeta'(0,x)$$

and

$$\zeta_3'(0,x)=\frac{1}{2}\zeta'(-2,x)+\frac{3-2x}{2}\zeta'(-1,x)+\frac{(x-1)(x-2)}{2}\zeta'(0,x)$$

Hence, letting $x=1$ and remarking $\zeta(s,1)=\zeta(s)$, we have

$$\Gamma_2(1)=\exp(\zeta_2'(0,1))=\exp(\zeta'(-1))$$

and

$$\Gamma_3(1)=\exp(\zeta_3'(0,1))=\exp\left(\frac{1}{2}\zeta'(-2)+\frac{1}{2}\zeta'(-1)\right).$$

Thus we have

$$\rho_2=\sqrt{2\pi}\exp(-\zeta'(-1))$$

and

$$\rho_3=\sqrt{2\pi}\exp\left(-\frac{3}{2}\zeta'(-1)-\frac{1}{2}\zeta'(-2)\right).$$

Now we use the functional equation for $\zeta(s)$:

$$\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

By differentiating both sides of this functional equation at $s = 2$, we have

$$\begin{aligned} -\zeta'(-1) &= 2(2\pi)^{-2}\Gamma(2)(\cos\pi)\zeta'(2) + 2(2\pi)^{-2}(-\log(2\pi))\Gamma(2)(\cos\pi)\zeta(2) \\ &\quad +2(2\pi)^{-2}\Gamma'(2)(\cos\pi)\zeta(2) \\ &= -\frac{\zeta'(2)}{2\pi^2} + \frac{\log(2\pi)}{12} - \frac{1-\gamma}{12}, \end{aligned}$$

where we used $\zeta(2) = \pi^2/6$ and $\Gamma'(2) = 1 - \gamma$. Hence

$$\begin{aligned} \rho_2 &= \sqrt{2\pi}\exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\log(2\pi)}{12} - \frac{1-\gamma}{12}\right) \\ &= \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{7\log(2\pi)+\gamma-1}{12}\right). \end{aligned}$$

Next, from the functional equation for $\zeta(s)$ we have

$$-\zeta'(-2) = 2(2\pi)^{-3}\Gamma(3)\left(-\frac{\pi}{2}\sin\frac{3\pi}{2}\right)\zeta(3).$$

Hence

$$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}.$$

Thus

$$\begin{aligned} \rho_3 &= \exp\left(\frac{\log(2\pi)}{2} - \frac{3}{2}\left(\frac{\zeta'(2)}{2\pi^2} - \frac{\log(2\pi)+\gamma-1}{12}\right) - \frac{1}{2}\left(-\frac{\zeta(3)}{4\pi^2}\right)\right) \\ &= \exp\left(\frac{\zeta(3)}{8\pi^2} - \frac{3\zeta'(2)}{4\pi^2} + \frac{5\log(2\pi)+\gamma-1}{8}\right). \end{aligned}$$

∎

3 Generalized Kummer's Formula

To prove Theorem 1 we show

Theorem 3 *Let $k \geq 1$ be an integer.*
(1) When k is odd,

$$\begin{aligned} \zeta'(-k, x) = \frac{2(-1)^{\frac{k+1}{2}}k!}{(2\pi)^{k+1}}\Bigg\{\sum_{n=1}^{\infty}\frac{(\log n)\cos(2\pi nx)}{n^{k+1}} \\ + \left(\log(2\pi) + \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{k+1}} \\ -\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{k+1}}\Bigg\}. \end{aligned}$$

(2) When k is even,

$$\zeta'(-k,x) = \frac{2(-1)^{\frac{k}{2}}k!}{(2\pi)^{k+1}} \Bigg\{ \sum_{n=1}^{\infty} \frac{(\log n)\sin(2\pi nx)}{n^{k+1}} + \left(\log(2\pi) + \gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{k+1}} + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{k+1}} \Bigg\}.$$

Proof. We have the following functional equation for $\zeta(s,x)$ proved by Hurwitz [8]:

$$\zeta(s,x) = 2(2\pi)^{s-1}\Gamma(1-s)\left(\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{1-s}}\right).$$

Hence we obtain

$$\begin{aligned}
\zeta'(-k,x) &= 2(2\pi)^{-k-1}(\log(2\pi)\cdot k! - \Gamma'(1+k)) \\
&\quad \times \left(-\sin\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{k+1}} + \cos\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{k+1}}\right) \\
&\quad + 2(2\pi)^{-k-1}k!\Bigg(-\sin\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{(\log n)\cos(2\pi nx)}{n^{k+1}} \\
&\quad + \cos\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{(\log n)\sin(2\pi nx)}{n^{k+1}} \\
&\quad + \frac{\pi}{2}\cos\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{k+1}} \\
&\quad + \frac{\pi}{2}\sin\left(\frac{\pi k}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{k+1}}\Bigg).
\end{aligned}$$

Thus, using

$$\frac{\Gamma'}{\Gamma}(k+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{k} - \gamma$$

we get (1) and (2). ∎

Proof of Theorem 1. From the proof of Theorem 2 we see the relations

$$\log\Gamma_2(x) = \zeta_2'(0,x) = \zeta'(-1,x) + (1-x)\zeta'(0,x) = \zeta'(-1,x) + (1-x)\log\Gamma_1(x)$$

and

$$\begin{aligned}
\log\Gamma_3(x) &= \zeta_3'(0,x) = \frac{1}{2}\zeta'(-2,x) + \frac{3-2x}{2}\zeta'(-1,x) + \frac{(x-1)(x-2)}{2}\zeta'(0,x) \\
&= \frac{1}{2}\zeta'(-2,x) + \left(\frac{3}{2}-x\right)\zeta_2'(0,x) - \frac{(x-1)^2}{2}\zeta'(0,x) \\
&= \frac{1}{2}\zeta'(-2,x) + \left(\frac{3}{2}-x\right)\log\Gamma_2(x) - \frac{(x-1)^2}{2}\log\Gamma_1(x).
\end{aligned}$$

Hence we have Theorem 1 from Theorem 3. ∎

4 Applications

As the first application of our generalization of Kummer's formula we calculate some special values $\Gamma_r(x)$ for $r = 2$ and 3. Before this we explain our method in the simple case $r = 1$: letting $x = 1/2$ in Kummer's formula (1.4) for $\Gamma_1(x)$ we have

$$\begin{aligned}
\Gamma_1\left(\frac{1}{2}\right) &= \exp\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\right) \\
&= \exp\left(-\frac{1}{2}\log 2\right) \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_1^B\left(\frac{1}{2}\right) &= \rho_1\Gamma_1\left(\frac{1}{2}\right) \\
&= \sqrt{2\pi}\cdot\frac{1}{\sqrt{2}} \\
&= \sqrt{\pi}.
\end{aligned}$$

(Actually in this case $r = 1$ these results are also direct from $\Gamma_1(x) = \Gamma(x)/\sqrt{2\pi}$ and $\Gamma_1^B(x) = \Gamma(x)$ using the well-known $\Gamma(1/2) = \sqrt{\pi}$.)

Theorem 4 *(1)*

$$\Gamma_2\left(\frac{1}{2}\right) = \exp\left(-\frac{\zeta'(2)}{4\pi^2} + \frac{\log\pi+\gamma-1}{24} - \frac{\log 2}{4}\right).$$

(2)

$$\Gamma_2^B\left(\frac{1}{2}\right) = \exp\left(-\frac{3\zeta'(2)}{4\pi^2} + \frac{5\log\pi+\gamma-1}{8} + \frac{\log 2}{3}\right).$$

(3)

$$\Gamma_3\left(\frac{1}{2}\right) = \exp\left(\frac{3\zeta(3)}{32\pi^2} - \frac{\zeta'(2)}{4\pi^2} + \frac{\log\pi+\gamma-1}{24} - \frac{3\log 2}{16}\right).$$

(4)

$$\Gamma_3^B\left(\frac{1}{2}\right) = \exp\left(\frac{7\zeta(3)}{32\pi^2} - \frac{\zeta'(2)}{\pi^2} + \frac{4\log\pi+\gamma-1}{6} + \frac{7\log 2}{16}\right).$$

Proof. (1). Let $x = 1/2$ in Theorem 1(1). Then we have

$$\log\Gamma_2\left(\frac{1}{2}\right) = -\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(\log n)\cos(\pi n)}{n^2} - \frac{\log(2\pi)+\gamma-1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^2} + \frac{1}{2}\log\Gamma_1\left(\frac{1}{2}\right).$$

We notice

$$\begin{aligned}\sum_{n=1}^{\infty}\cos(\pi n)n^{-s} &= -\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} \\ &= -(1-2\cdot 2^{-s})\zeta(s) \\ &= (2^{1-s}-1)\zeta(s)\end{aligned}$$

and its differentiation

$$\sum_{n=1}^{\infty}(\log n)\cos(\pi n)n^{-s} = (\log 2)2^{1-s}\zeta(s) - (2^{1-s}-1)\zeta'(s).$$

Hence

$$\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^2} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\log n)\cos(\pi n)}{n^2} &= \frac{\log 2}{2}\zeta(2) + \frac{1}{2}\zeta'(2) \\ &= \frac{\pi^2\log 2}{12} + \frac{\zeta'(2)}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\log\Gamma_2\left(\frac{1}{2}\right) &= -\frac{\log 2}{24} - \frac{\zeta'(2)}{4\pi^2} + \frac{\log(2\pi)+\gamma-1}{24} - \frac{\log 2}{4} \\ &= -\frac{\zeta'(2)}{4\pi^2} + \frac{\log\pi+\gamma-1}{24} - \frac{\log 2}{4}.\end{aligned}$$

This gives (1).

(2). Since $\Gamma_2^B(1/2) = \rho_2\Gamma_2(1/2)$, (2) follows from (1) and the formula for ρ_2 in Theorem 2.

(3). Set $x = 1/2$ in Theorem 1(2). Then

$$\begin{aligned}\log\Gamma_3\left(\frac{1}{2}\right) &= -\frac{1}{8\pi^2}\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3} + \log\Gamma_2\left(\frac{1}{2}\right) - \frac{1}{8}\log\Gamma_1\left(\frac{1}{2}\right) \\ &= \frac{3\zeta(3)}{32\pi^2} + \log\Gamma_2\left(\frac{1}{2}\right) + \frac{\log 2}{16},\end{aligned}$$

where we used

$$\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3} = (2^{1-3}-1)\zeta(3) = -\frac{3}{4}\zeta(3)$$

and $\Gamma_1(1/2) = 1/\sqrt{2}$. Hence we obtain (3) from (1).

(4). Since $\Gamma_3^B(1/2) = \rho_3\Gamma_3(1/2)$, (4) follows from (3) and the fomula for ρ_3 in Theorem 2. ∎

Values of $\Gamma_r(x)$ at rational numbers are also expressed via special values of Dirichlet L-functions besides $\zeta(s)$ through the generalized Kummer's formula. Here we report the following typical example.

Theorem 5

$$\Gamma_2\left(\frac{1}{4}\right) = \exp\left(\frac{L(2,\chi_{-4})}{4\pi} - \frac{3L'(1,\chi_{-4})}{4\pi} - \frac{\zeta'(2)}{16\pi^2} + \frac{19\log\pi + 19\gamma + \log 2 - 1}{96}\right),$$

where $L(s,\chi_{-4})$ is the Dirichlet L-function for the non-trivial character χ_{-4} modulo 4.

Proof. Let $x = 1/4$ in Theorem 1(1). Then we have

$$\begin{aligned}\log\Gamma_2\left(\frac{1}{4}\right) = &-\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(\log n)\cos(\frac{\pi n}{2})}{n^2} - \frac{\log(2\pi)+\gamma-1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(\frac{\pi n}{2})}{n^2} \\ &+ \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{\sin(\frac{\pi n}{2})}{n^2} + \frac{3}{4}\log\Gamma_1\left(\frac{1}{4}\right).\end{aligned}$$

We notice

$$\sum_{n=1}^{\infty}\frac{\cos(\frac{\pi n}{2})}{n^s} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^s} = 2^{-s}(2^{1-s}-1)\zeta(s) = (2^{1-2s}-2^{-s})\zeta(s)$$

and its differentiation

$$\sum_{n=1}^{\infty}\frac{(\log n)\cos(\frac{\pi n}{2})}{n^s} = -(2^{1-2s}-2^{-s})\zeta'(s) + (2^{-s}-2^{2-2s})(\log 2)\zeta(s).$$

Hence

$$\sum_{n=1}^{\infty} \frac{\cos(\frac{\pi n}{2})}{n^2} = -\frac{\zeta(2)}{8} = -\frac{\pi^2}{48}$$

and

$$\sum_{n=1}^{\infty} \frac{(\log n)\cos(\frac{\pi n}{2})}{n^2} = \frac{\zeta'(2)}{8}.$$

Using

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{2})}{n^s} = \sum_{\substack{n=1\\ n:\text{odd}}}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^s} = L(s,\chi_{-4})$$

we have

$$\Gamma_2\left(\frac{1}{4}\right) = -\frac{\zeta'(2)}{16\pi^2} + \frac{\log(2\pi)+\gamma-1}{96} + \frac{L(2,\chi_{-4})}{4\pi} + \frac{3}{4}\log\Gamma_1\left(\frac{1}{4}\right).$$

Thus we obtain Theorem 5 from the result of Malmstén [20]

$$\log\Gamma_1\left(\frac{1}{4}\right) = -\frac{L'(1,\chi_{-4})}{\pi} + \frac{\log\pi+\gamma}{4},$$

which is obtained by setting $x = 1/4$ in Kummer's formula (1.4):

$$\begin{aligned} \log\Gamma_1\left(\frac{1}{4}\right) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\log n)\sin(\frac{\pi n}{2})}{n} \\ &\quad + \frac{\log(2\pi)+\gamma}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\frac{\pi n}{2})}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos(\frac{\pi n}{2})}{n} && (4.1) \\ &= -\frac{1}{\pi}L'(1,\chi_{-4}) + \frac{\log(2\pi)+\gamma}{\pi}L(1,\chi_{-4}) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n} && (4.2) \\ &= -\frac{1}{\pi}L'(1,\chi_{-4}) + \frac{\log\pi+\gamma}{4}, && (4.3) \end{aligned}$$

where we used $L(1,\chi_{-4}) = \pi/4$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \log 2$. ∎

Now let $S_r(x) = \Gamma_r(x)^{-1}\Gamma_r(r-x)^{(-1)^r}$ be the multiple sine function studied in [13]. As the second application of the generalized Kummer's formula we prove basic properties of $S_r(x)$ for $r = 2$ and 3. This generalizes the usual sine function:

$$S_1(x) = \Gamma_1(x)^{-1}\Gamma_1(1-x)^{-1} = \frac{2\pi}{\Gamma(x)\Gamma(1-x)} = 2\sin(\pi x).$$

We notice that

$$S_1(x) = \exp\left(-\sum_{n=1}^{\infty}\frac{\cos(2\pi n x)}{n}\right)$$

for $0 < x < 1$ from Kummer's formula for $\Gamma_1(x)$.

Theorem 6 *(1)*

$$\begin{aligned} S_2(x) &= \exp\left(-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} + (x-1)\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}\right) \\ &= \exp\left(-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2}\right)\cdot S_1(x)^{1-x} \end{aligned}$$

for $0 < x < 1$.

(2)

$$\frac{S_2'}{S_2}(x) = -\pi(x-1)\cot(\pi x).$$

(3)

$$S_2\left(\frac{1}{2}\right) = \sqrt{2}.$$

(4)

$$S_2(x) = 2\pi x e^{-x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)^{1+n}\left(1-\frac{x}{n}\right)^{1-n}e^{-2x}\right).$$

(5)

$$S_2(x) = \exp\left(-\pi\int_0^{x-1} t\cot(\pi t)dt\right).$$

Proof. Since

$$S_2(x) = \Gamma_2(x)^{-1}\Gamma_2(2-x)$$

and

$$\Gamma_2(2-x) = \Gamma_2(1-x)\Gamma_1(1-x)^{-1},$$

we have

$$S_2(x) = \Gamma_2(x)^{-1}\Gamma_2(1-x)\Gamma_1(1-x)^{-1}.$$

Hence, from (1) of Theorem 1 we have

$$\log S_2(x) = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} - (x-1)\log S_1(x)$$

and

$$\begin{aligned} \frac{S_2'}{S_2}(x) &= -(x-1)\frac{S_1'}{S_1}(x) - \sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n} - \log S_1(x) \\ &= -(x-1)\frac{S_1'}{S_1}(x) \\ &= -(x-1)\pi\cot(\pi x), \end{aligned}$$

where we used

$$\begin{aligned}\log S_1(x) &= -\log\Gamma_1(x)-\log\Gamma_1(1-x)\\ &= -\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}.\end{aligned}$$

Thus (1) and (2) are proved. We have (3) by setting $x=1/2$ in (1). For the proof of (4) let

$$\mathcal{S}_2(z)=e^z\prod_{n=1}^{\infty}\left(\left(\frac{1-\frac{z}{n}}{1+\frac{z}{n}}\right)^n e^{2z}\right)$$

following Hölder [7]. Then we easily obtain

$$\frac{\mathcal{S}_2'}{\mathcal{S}_2}(x)=\pi x\cot(\pi x)$$

and $\mathcal{S}_2(1/2)=\sqrt{2}$ ([7, 13]). Hence it is sufficient to show that

$$S_2(x)=\mathcal{S}_2(x)^{-1}S_1(x).$$

We know $S_2(1/2)=\sqrt{2}$ and $S_1(1/2)=2$. Thus the both sides are $\sqrt{2}$ at $x=1/2$, and the differentiations of the both sides turn out to be $-(x-1)\cot(\pi x)$. Hence we get (4). Lastly (5) follows from (2) since both sides are 1 at $x=1$. ∎

Remark 1 These results have further applications containing expressions of the difficult special values of Dirichlet L-functions ([13]). Especially (5) gives the calculation of the gamma factor of the Selberg zeta function for a Riemann surface (Sarnak [23] and Voros [25]). The situation is similar in the higher dimensional case, where we use the multiple sine function $S_r(x)$ with r being the dimension ([13]).

Theorem 7 *(1)*

$$\begin{aligned}S_3(x) &= \exp\left(\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3}-\frac{2x-3}{4\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2}-\frac{(x-1)(x-2)}{2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}\right)\\ &= \exp\left(\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3}\right)S_2(x)^{\frac{3}{2}-x}S_1(x)^{-\frac{(x-1)^2}{2}}\end{aligned}$$

for $0<x<1$.

(2)

$$\frac{S_3'}{S_3}(x)=\frac{\pi}{2}(x-1)(x-2)\cot(\pi x).$$

(3)

$$S_3\left(\frac{1}{2}\right) = 2^{\frac{3}{8}} \exp\left(-\frac{3}{16\pi^2}\zeta(3)\right).$$

(4)

$$S_3(x) = 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4}-\frac{3}{2}x} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{n}\right)^{\frac{(n+1)(n+2)}{2}} \left(1-\frac{x}{n}\right)^{\frac{(n-1)(n-2)}{2}} e^{\frac{x^2}{2}-3x} \right).$$

Proof. Using $\Gamma_r(x) = \Gamma_r(x-1)\Gamma_{r-1}(x-1)^{-1}$ we have

$$\begin{aligned} S_3(x) &= \Gamma_3(x)^{-1}\Gamma_3(3-x)^{-1} \\ &= \Gamma_3(x)^{-1}(\Gamma_3(2-x)\Gamma_2(2-x)^{-1})^{-1} \\ &= \Gamma_3(x)^{-1}(\Gamma_3(1-x)\Gamma_2(1-x)^{-1}\Gamma_2(1-x)^{-1}\Gamma_1(1-x))^{-1} \\ &= \Gamma_3(x)^{-1}\Gamma_3(1-x)^{-1}\Gamma_2(1-x)^2\Gamma_1(1-x)^{-1}. \end{aligned}$$

Hence, from Theorem 1(2), for $0 < x < 1$

$$\begin{aligned} \log S_3(x) &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} - \left(\frac{3}{2}-x\right)\log\Gamma_2(x) + \frac{(x-1)^2}{2}\log\Gamma_1(x) \\ &\quad - \left(\frac{1}{2}+x\right)\log\Gamma_2(1-x) + \frac{x^2}{2}\log\Gamma_1(1-x) + 2\log\Gamma_2(1-x) - \log\Gamma_1(1-x) \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} + \left(\frac{3}{2}-x\right)\log(\Gamma_2(1-x)\Gamma_2(x)^{-1}) \\ &\quad + \frac{(x-1)^2}{2}\log\Gamma_1(x) + \left(\frac{x^2}{2}-1\right)\log\Gamma_1(1-x) \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} + \left(\frac{3}{2}-x\right)\log(S_2(x)\Gamma_1(1-x)) \\ &\quad + \frac{(x-1)^2}{2}\log\Gamma_1(x) + \left(\frac{x^2}{2}-1\right)\log\Gamma_1(1-x) \\ &= \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^3} + \left(\frac{3}{2}-x\right)\log S_2(x) - \frac{(x-1)^2}{2}\log S_1(x), \end{aligned}$$

where we used that

$$\Gamma_2(1-x)\Gamma_2(x)^{-1} = S_2(x)\Gamma_1(1-x)$$

and

$$\Gamma_1(x)^{-1}\Gamma_1(1-x)^{-1} = S_1(x).$$

(2). From (1) we have

$$\frac{S_3'}{S_3}(x) = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} - (\log S_2(x) + (x-1)\log S_1(x)) + \left(\left(\frac{3}{2}-x\right)\frac{S_2'}{S_2}(x) - \frac{(x-1)^2}{2}\frac{S_1'}{S_1}(x)\right).$$

Hence, using the previous formulas for $\log S_r(x)$ and $S_r'(x)/S_r(x)$ $(r=1,2)$, we obtain

$$\begin{aligned}\frac{S_3'}{S_3}(x) &= -\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} \\ &\quad +\pi\left((x-1)\left(x-\frac{3}{2}\right) - \frac{(x-1)^2}{2}\right)\cot(\pi x) \\ &= \frac{\pi}{2}(x-1)(x-2)\cot(\pi x).\end{aligned}$$

(3). From (1)

$$\begin{aligned}S_3\left(\frac{1}{2}\right) &= \exp\left(\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(\pi nx)}{n^3}\right) S_2\left(\frac{1}{2}\right) S_1\left(\frac{1}{2}\right)^{-\frac{1}{8}} \\ &= 2^{\frac{3}{8}}\exp\left(\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\right).\end{aligned}$$

Here

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} &= -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} \\ &= -(1-2\cdot 2^{-3})\zeta(3) \\ &= -\frac{3}{4}\zeta(3).\end{aligned}$$

Hence we obtain (3).

(4). Let

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}}\prod_{n=1}^{\infty}\left(\left(1-\frac{x^2}{n^2}\right)^{n^2} e^{x^2}\right).$$

We prove that

$$\mathcal{S}_3(x) = C\cdot S_3(x)^2 S_2(x)^{-3} S_1(x)$$

with $C = e^{2\zeta'(-2)}$. Then we get (4) using the product expressions for $\mathcal{S}_3(x)$, $S_2(x)$ and $S_1(x)$. First, we see

$$\frac{\mathcal{S}_3'}{\mathcal{S}_3}(x) = \pi x^2\cot(\pi x)$$

from [12, 13, 21] and

$$\begin{aligned} 2\frac{S_3'}{S_3}(x) - 3\frac{S_2'}{S_2}(x) + \frac{S_1'}{S_1}(x) &= \pi\left(2\cdot\frac{(x-1)(x-2)}{2} + 3(x-1) + 1\right)\cot(\pi x) \\ &= \pi x^2\cot(\pi x). \end{aligned}$$

Hence, it is sufficient to show that

$$C = \mathcal{S}_3\left(\frac{1}{2}\right) S_3\left(\frac{1}{2}\right)^{-2} S_2\left(\frac{1}{2}\right)^{3} S_1\left(\frac{1}{2}\right)^{-1}.$$

We calculated the value $\mathcal{S}_3\left(\frac{1}{2}\right)$ in [16] ([13]) as

$$\mathcal{S}_3\left(\frac{1}{2}\right) = 2^{\frac{1}{4}}\exp\left(-\frac{7}{8\pi^2}\zeta(3)\right),$$

which is equivalent to the following result of Euler [3]:

$$\zeta(3) = \frac{2\pi^2}{7}\log 2 + \frac{16}{7}\int_0^{\frac{\pi}{2}} x\log(\sin x)dx.$$

We know

$$S_3\left(\frac{1}{2}\right) = 2^{\frac{3}{8}}\exp\left(-\frac{3}{16\pi^2}\zeta(3)\right)$$

as above. Thus we have

$$\begin{aligned} &\mathcal{S}_3\left(\frac{1}{2}\right) S_3\left(\frac{1}{2}\right)^{-2} S_2\left(\frac{1}{2}\right)^{3} S_1\left(\frac{1}{2}\right)^{-1} \\ &= 2^{\frac{1}{4}}\exp\left(-\frac{7}{8\pi^2}\zeta(3)\right)\left(2^{\frac{3}{8}}\exp\left(-\frac{3}{16\pi^2}\zeta(3)\right)\right)^{-2}(\sqrt{2})^3\cdot 2^{-1} \\ &= \exp\left(-\frac{\zeta(3)}{2\pi^2}\right) \\ &= \exp(2\zeta'(-2)) \\ &= C, \end{aligned}$$

where we used $\zeta(3) = -4\pi^2\zeta'(-2)$ coming from the functional equation for $\zeta(s)$. ∎

Remark 2 For another approach to Theorems 6 and 7 we refer to [13], where we use multiplication formulas for multiple sine functions and the argument is more elaborate. The above proofs are quite direct because of our Kummer's formula for multiple gamma functions.

5 Generalizations and Problems

Our investigation naturally leads to the case of $\Gamma_r(x)$ for $r \geqq 4$. This is treated similarly as $r = 2$ and 3 by using

$$\zeta_r(s,x) = \frac{1}{(r-1)!}\zeta(s-r+1,x) + \sum_{k=1}^{r-1} b_{r,k}(x)\zeta_k(s,x)$$

where $b_{r,k}(x)$ is a polynomial in x determined by

$$\binom{n+r-1}{r-1} = \frac{(n+x)^{r-1}}{(r-1)!} + \sum_{k=1}^{r-1} b_{r,k}(x)\binom{n+k-1}{k-1}$$

for all integers $n \geqq 0$. For example

$$b_{r,1}(x) = -\frac{(x-1)^{r-1}}{(r-1)!}.$$

Then we have

$$\begin{aligned}\Gamma_r(x) &= \exp\left(\zeta_r'(0,x)\right) \\ &= \exp\left(\frac{1}{(r-1)!}\zeta'(1-r,x)\right)\prod_{k=1}^{r-1}\Gamma_k(x)^{b_{r,k}(x)}.\end{aligned}$$

Hence our Theorem 3 gives the Kummer's formula for $\Gamma_r(x)$ inductively.

Beyond this we have a more general problem for $\Gamma_r(x;(\omega_1,\cdots,\omega_r))$ with general periods $(\omega_1,\cdots,\omega_r)$ defined by

$$\Gamma_r(x;(\omega_1,\cdots,\omega_r)) = \exp\left(\zeta_r'(0,x,(\omega_1,\cdots,\omega_r))\right),$$

where

$$\zeta_r(s,x,(\omega_1,\cdots,\omega_r)) = \sum_{n_1,\ldots,n_r=0}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r+x)^{-s}$$

is the multiple Hurwitz zeta function for general parameters. We have seen the case $(\omega_1,\cdots,\omega_r) = (1,...,1)$ above. First, when $(\omega_1,\cdots,\omega_r)$ is reduced to the rational parameters the situation is quite similar to the case $(1,...,1)$. For example, if $(\omega_1,\omega_2) = (1,2)$, then

$$\zeta_2(s,x,(1,2)) = 2^{-s}\left(\zeta_2\left(s,\frac{x}{2}\right) + \zeta_2\left(s,\frac{x+1}{2}\right)\right)$$

and

$$\zeta_2'(0,x,(1,2)) = \zeta_2'\left(0,\frac{x}{2}\right) + \zeta_2'\left(0,\frac{x+1}{2}\right) - (\log 2)\left(\zeta_2\left(0,\frac{x}{2}\right) + \zeta_2\left(0,\frac{x+1}{2}\right)\right)$$

(see [15]). Since

$$\zeta_2\left(0,\frac{x}{2}\right)+\zeta_2\left(0,\frac{x+1}{2}\right) = \zeta_2,(0,x,(1,2)) = \frac{1}{4}\left(x^2-3x+\frac{11}{6}\right),$$

we have

$$\log\Gamma_2(x,(1,2)) = \log\Gamma_2\left(\frac{x}{2}\right)+\log\Gamma_2\left(\frac{x+1}{2}\right)-\frac{\log 2}{4}\left(x^2-3x+\frac{11}{6}\right).$$

Hence we obtain Kummer's formula for $\Gamma_2(x,(1,2))$ from our Theorem 1(1).

The non-rational (or "non-commensurable") parameter case, we face the problem to have the functional equation for $\zeta_r(s,x,(\omega_1,\cdots,\omega_r))$ of the form

$$\zeta_r(s,x,(\omega_1,\cdots,\omega_r)) = \xi_r(r-s,x,(\omega_1,\cdots,\omega_r))$$

by using the residue calculation as in Riemann [22] and Hurwitz [8]. This problem is in general highly non-trivial (delicate convergence) as investigated by Hardy [5, 6]. After that we would have the desired Kummer's type formula:

$$\log\Gamma_r(x;(\omega_1,\cdots,\omega_r)) = -\xi_r'(r,x,(\omega_1,\cdots,\omega_r)).$$

We postpone the detailed investigation to the next opportunity.

References

[1] E.W. Barnes: "On the theory of the multiple gamma function. Trans. Cambridge Philos. Soc., **19** (1904) 374-425

[2] S. Chowla and A. Selberg "On Epstein's zeta-function" J. reine angew. Math. (Crelle's J.) **227** (1967) 86-110.

[3] L. Euler: "Exercitationes analyticae" Novi Commentarii Academiae Scientiarum Petropolitanae, **17** (1772) 173-204. (Opera Omnia I-15, pp. 131-167)

[4] G.H. Hardy "On Kummer's series for $\log\Gamma(a)$" Quarterly J. Math. **37** (1906) 49-53. (Collected Papers: Vol IV, pp. 428-432)

[5] G.H. Hardy "The expression of the double zeta-function and double gamma-function in terms of elliptic functions" Trans. Cambridge Phil. Soc. **20** (1905) 1-35. (Collected Papers: Vol. IV, pp. 393-427)

[6] G.H. Hardy "On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters" Quarterly J. Math. **37** (1906) 53-79. (Collected Papers: Vol. IV, pp. 433-459)

[7] O. Hölder: "Ueber eine transcendente Function" Göttingen Nachrichten 1886, Nr. 16. pp. 514-522.

[8] A. Hurwitz "Einige Eigenschaften der Dirichlet'schen Funktionen $F(s) = \sum(\frac{D}{n}) \cdot \frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen binärer quadratischer Formen auftreten" Zeitschrift für Mathematik und Physik **27** (1882) 86-101. (Werke Vol. I, pp. 72-88)

[9] E. Kummer "Beitrag zur Theorie der Function $\Gamma(x) = \int_0^\infty e^{-v}v^{x-1}dv$" J. reine angew. Math. (Crelle's J.) **35** (1847) 1-4. (Collected Papers Vol. II, pp.325-328)

[10] N. Kurokawa: "Multiple sine functions and Selberg zeta functions" Proc. Japan Acad. **67A** (1991) 61-64.

[11] N. Kurokawa: "Gamma factors and Plancherel measures" Proc. Japan Acad. **68A** (1992) 256-260.

[12] N. Kurokawa: "Multiple zeta functions: an example" In Zeta Functions in Geometry, volume 21 of Advanced Studies in Pure Math., pages 219-226, Kinokuniya, Tokyo 1992.

[13] N. Kurokawa and S. Koyama: "Multiple sine functions" Forum Math. (in press)

[14] S. Koyama and N. Kurokawa: "Multiple zeta functions I" Composit. Math. (in press)

[15] S. Koyama and N. Kurokawa: "Normalized double sine functions" Proc. Japan Acad. **79** (2003) 14-18.

[16] N. Kurokawa and M. Wakayama: "On $\zeta(3)$" J. Ramanujan Math. Soc. **16** (2001) 205-214.

[17] E. Landau "Über die zu einem algebraischen Zahlkörper gehörige Zetafunction und die Ausdehnung der Tschebyschefschen Primzahltheorie auf das Problem der Verteilung der Primideale" J. reine angew. Math. (Crelle's J.) **125** (1902) 64-188. (Collected Works Vol. I, pp. 201-326)

[18] M. Lerch "Dalši studie v oboru Malmsténovských řad" Rozpravy České Akad. **3** (1894) no. 28, pp. 1-61.

[19] M. Lerch "Sur quelques formules relatives du nombre des classes" Bull. Sci. Math. **21** (1897) 290-304.

[20] C.J. Malmstén "De integralibus quibusdam definitis, seriebusque infinitis" J. reine angew. Math. (Crelle's J.) **38** (1849) 1-39.

[21] Yu. I. Manin: "Lectures on zeta functions and motives (according to Deninger and Kurokawa)" Asterisque **228** (1995) 121-163.

[22] B. Riemann: "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" Monatsberichte der Berliner Akademie, November 1859, pp. 671-680. (Gesammelte Mathematische Werke, pp. 145-155)

[23] P. Sarnak: "Determinants of Laplacians" Comm. Math. Phys. **110** (1987) 113–120.

[24] T. Shintani: "On a Kronecker limit formula for real quadratic fields" J. Fac. Sci. Univ. Tokyo, **24** (1977) 167-199.

[25] A. Voros: "Spectral functions, special functions and Selberg trace formula" Comm. Math. Phys. **110** (1987) 439-465.

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