Subconvexity of Hecke L-functions in the Grossencharacter-aspect

Shin-ya Koyama

1. Introduction. Let K be a number field with $[K:\mathbf{Q}]=n=r_1+2r_2$. The real and complex conjugates of K are denoted by $K^{(q)}$ $(q=1,...,r_1)$ and $K^{(p)}$ $(p=r_1+1,...,n)$ with $K^{(p+r_2)}=\overline{K^{(p)}}$. The conjugate of $\mu\in K$ in $K^{(j)}$ $(j=1,...,n)$ is denoted by $\mu^{(j)}$. Put $N(\mu)=\mu^{(1)}\cdots\mu^{(n)}$.

Let $\mathbf{J}_K$ and $\mathbf{P}_K$ be the idele group and the principal idele group of K, respectively. Grossencharacter λ of K is a continuous character of the idele class group $\mathbf{C}_K=\mathbf{J}_K/\mathbf{P}_K$. We use the same symbol λ for the induced character of $\mathbf{J}_K$. We decompose as $\mathbf{J}_K=\mathbf{J}_\infty\times\mathbf{J}_0$ with the infinite part $\mathbf{J}_\infty$ and the finite part $\mathbf{J}_0$. Let $\mathbf{U}_0$ be the unit group of $\mathbf{J}_0$, and put $\mathbf{U}_{\mathfrak{m},0}=\{\mathfrak{u}\in\mathbf{U}_0\mid\mathfrak{u}\equiv 1\pmod{\mathfrak{m}}\}$, where $\mathfrak{m}$ is an integral ideal of K. The set $\{\mathbf{U}_{\mathfrak{m},0}\}$ is a fundamental neighbourhood system of 1 in $\mathbf{J}_0$. A map from $\mathbf{J}_{\mathfrak{m},0}=\{\mathfrak{a}\in\mathbf{J}_0\mid\mathfrak{a}_\mathfrak{p}=1\ \ (\forall\mathfrak{p}|\mathfrak{m})\}$ to an ideal class group $G(\mathfrak{m})=\{\tilde{\mathfrak{a}}\mid(\mathfrak{a},\mathfrak{m})=1\}$ defined by

$$\mathbf{J}_{\mathfrak{m},0}\ni\mathfrak{a}\mapsto\tilde{\mathfrak{a}}=\prod_{\mathfrak{p}}\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}\in G(\mathfrak{m})$$

is a homomorphism whose kernel is a subset of $\mathbf{U}_{\mathfrak{m},0}$.

By continuity a Grossencharacter λ satisfies that $\lambda(\mathbf{U}_{\mathfrak{m},0})=1$ for some $\mathfrak{m}$. The greatest common divisor $\mathfrak{f}$ of such $\mathfrak{m}$'s are called the conductor of λ. For $\mathfrak{a}\in\mathbf{J}_{\mathfrak{f},0}$, the value $\lambda(\mathfrak{a})$ is determined by the ideal class $\tilde{\mathfrak{a}}\in G(\mathfrak{f})$. Thus we define a character $\tilde\lambda$ of $G(\mathfrak{f})$ as $\tilde\lambda(\tilde{\mathfrak{a}}):=\lambda(\mathfrak{a})$. The Hecke L-function with respect to the Grossencharacter λ is defined by

$$L(s,\lambda)=\sum_{\tilde{\mathfrak{a}}\in G(\mathfrak{f})}\frac{\tilde\lambda(\tilde{\mathfrak{a}})}{N(\tilde{\mathfrak{a}})^s}.\quad(\mathrm{Re}(s)>1)$$

When we restrict λ on $\mathbf{J}_\infty=\mathbf{R}^{\times r_1}\times\mathbf{C}^{\times r_2}$, we have for $x=(x^{(1)},...,x^{(r_1+r_2)})\in\mathbf{J}_\infty$

$$\lambda(x)=\prod_{v=1}^{r_1}(\mathrm{sgn}\ x^{(v)})^{a_v}\prod_{v=1}^{r_1+r_2}|x^{(v)}|^{ib_v}\prod_{v=r_1+1}^{r_1+r_2}\left(\frac{x^{(v)}}{|x^{(v)}|}\right)^{a_v}$$

with $a_v\in\mathbf{Z}$ and $b_v\in\mathbf{R}$, and $a_1,...,a_{r_1}$, which may take only the values 0 or 1, depend only on the narrow class character mod $\mathfrak{f}$.

In what follows we fix $a_1,...,a_{r_1}$ so that the narrow class character χ mod $\mathfrak{f}$ is fixed. We will make the remaining part in the Grossencharacter vary, which is parametarized by an n-dimensional vector $\mathbf{a}=(b_1,...,b_{r_1+r_2},a_{r_1+1},...,a_{r_1+r_2})$ with $\sum_v b_v=0$. We denote such Hecke character λ by $\chi\lambda^{\mathbf{a}}$. If $\chi\lambda^{\mathbf{a}}$ is a normalized Hecke character, then $\chi\lambda^{m\mathbf{a}}$ is also normalized for any $m\in\mathbf{Z}$. The chief concern of this paper is to estimate the size of $L(\frac{1}{2}+it, \chi\lambda^{m\mathbf{a}})$ as $|m| \to \infty$ with χ, $\mathbf{a}$ and t being fixed. We may call this problem the m-aspect in the estimate of Hecke L-functions. In what follows we write $\chi\lambda^m := \chi\lambda^{m\mathbf{a}}$.

Duke [D] computed the convexity bound

$$L(\frac{1}{2}+it, \chi\lambda^m) \ll_{t,\varepsilon} |m|^{\frac{n}{4}+\varepsilon} \tag{1.1}$$

which can be proved by applying the Phragmén-Lindelöf principle to the lines $\mathrm{Re}(s) = 1+\delta$ and $\mathrm{Re}(s) = -\delta$ with $\delta > 0$. We improve this bound by using the approximate functional equation in the m-aspect, which will be proved in the next section. In Section 3 we express the exponential sum over certain range of norms in terms of the sum over algebraic integers in certain cubes. The main theorem is proved using the van der Corput method in Section 4. It asserts that

Theorem 1.1. *In the above notation we have as* $|m| \to \infty$

$$L(\frac{1}{2}+it, \lambda^{m\mathbf{a}}) \ll_{t,\varepsilon} |m|^{\frac{n}{4}-\frac{1}{18}+\varepsilon}$$

for any $\varepsilon > 0$.

The Grossencharacter-aspect is recently considered in a different context. In their paper [PS], Petridis and Sarnak obtain a subconvexity estimate of automorphic L-functions $L(s,\phi)$ for a Maass cusp form ϕ of $SL(2, Z[i])$. In the proof they consider the twists with Grossencharacters and take an avarage in the form

$$\sum \int |L(\frac{1}{2}+it, \phi\otimes\lambda^m)|^2 dt,$$

where the summation and the integration are taken over certain range of (m,t). Consequently they succeed in obtaining the subconvexities in the both m and t aspects.

2. Approximate Functional Equation. We require the functional equation for the Hecke L-function given by Mordell [Mo] (5.14):

$$L(s,\chi\lambda) = W(\chi\lambda)A^{1-2s}G(1-s,\hat{\chi}\bar{\lambda})L(1-s,\bar{\chi}\bar{\lambda}), \tag{2.1}$$

where $|W(\chi\lambda)| = 1$ and $A = \left(\frac{d_K N(\mathfrak{f})}{\pi^n 2^{2r_2}}\right)^{1/2}$ with d_K the discriminant of K, $\hat{\chi}$ is the real character induced by χ on the group of narrow ideal classes lying over the broad class of principal ideals, and $G(s,\hat{\chi}\lambda)$ is defined by

$$G(s,\hat{\chi}\lambda) = \prod_{v=1}^{r_1} \frac{\Gamma(\frac{1}{2}(s+a_v+ib_v))}{\Gamma(\frac{1}{2}(1-s+a_v-ib_v))} \prod_{v=r_1+1}^{r_1+r_2} \frac{\Gamma(s+\frac{1}{2}|a_v|+ib_v)}{\Gamma(1-s+\frac{1}{2}|a_v|-ib_v)}.$$

We denote each factor by G_v, namely, $G(s,\hat{\chi}\lambda) = \prod_{v=1}^{r_1+r_2} G_v(s,\hat{\chi}\lambda)$.

Lemma 2.1. *The function $G(s, \hat{\chi}\lambda)$ is holomorphic except at the following simple poles:*

$$s = -2k - a_v - ib_v \quad (1 \le v \le r_1)$$

$$\text{with residue} \quad \frac{(-1)^k 2^{2(k+a_v)+1}(k+a_v)!}{\sqrt{\pi}k!(2(k+a_v))!} \prod_{v' \ne v} G_{v'}(-2k - a_v - ib_v)$$

$$s = -k - \frac{|a_v|}{2} - ib_v \quad (r_1 < v \le r_1 + r_2)$$

$$\text{with residue} \quad \frac{(-1)^k}{k!(k+|a_v|)!} \prod_{v' \ne v} G_{v'}(-k - \frac{|a_v|}{2} - ib_v)$$

with $k \ge 0$, $k \in \mathbf{Z}$, and the sum $\sum_\rho \frac{\mathrm{Res}_{s=\rho} G(s,\hat{\chi}\lambda)}{\rho}$ over all poles of $G(s, \hat{\chi}\lambda)$ absolutely converges. In particular, $G(s, \hat{\chi}\lambda)$ is holomorphic in $\mathrm{Re}(s) > 0$.

Proof. We first treat the case $1 \le v \le r_1$. The numerator of

$$G_v(s, \hat{\chi}\lambda) = \frac{\Gamma(\frac{1}{2}(s + a_v + ib_v))}{\Gamma(\frac{1}{2}(1 - s + a_v - ib_v))}$$

has poles at $s = -2k - a_v - ib_v$, none of which agrees to any zero coming from a pole of the denominator, since $k \ge 0$. By the fact that the residue of $\Gamma(z)$ at $z = -k$ is $\frac{(-1)^k}{k!}$, the residue of $G_v(s, \hat{\chi}\lambda)$ at $s = -2k - a_v - ib_v$ is computed as

$$\frac{2(-1)^k}{k!\Gamma\left(\frac{1+2k+2a_v}{2}\right)} = \frac{(-1)^k 2^{2(k+a_v)+1}(k+a_v)!}{\sqrt{\pi}k!(2k+2a_v)!},$$

where we used the identity

$$\Gamma\left(N + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2N)!}{2^{2N}N!}$$

for any nonnegative integer N. The case $r_1 + 1 \le v \le r_1 + r_2$ can be treated similarly.

Huxley's expression of $G_v(s, \hat{\chi}\lambda)$ (See (2.7) below) and Stirling's formula show that $G_{v'}(-2k - a_v - ib_v)$ and $G_{v'}(-k - \frac{|a_v|}{2} - ib_v)$ decay exponentially as $k \to \infty$. It gives the absolute convergence of the sum. □

In what follows we define for $z \in \mathbf{C} \setminus \mathbf{R}^-$

$$z^z := \exp(z \log z) = \exp(z(\log|z| + i\arg(z)))$$

with $|\arg(z)| < \pi$.

Lemma 2.2. *Put $z = s + iy \in \mathbf{C}$ with $s = \sigma + it$, $\sigma, t \in \mathbf{R}$ being fixed. Then we have*

$$z^z = (s+iy)^{s+iy} = |y|^{s+iy} \exp\left(s + \operatorname{sgn}(y)\frac{\pi i z}{2}\right)\left(1 + O\left(\frac{1}{y}\right)\right)$$

as $|y| \to \infty$.

Proof. Putting $|z| = r$ and $\arg(z) = \theta$ give

$$z^z = r^\sigma \exp(-\theta(t+y) + i((t+y)\log r + \theta\sigma)).$$

By putting $\varphi = \frac{\pi}{2} - |\theta| = \frac{\pi}{2} - \operatorname{sgn}(t+y)\theta$, we have $\sin\varphi = \varphi + O(\varphi^3)$. On the other hand we have $\sin\varphi = \frac{\sigma}{\sqrt{\sigma^2+y^2}} = \frac{\sigma}{|y|} + O(\frac{1}{y^2})$. Thus $\theta(t+y) \geq 0$, and

$$\varphi = \frac{\sigma}{|y|} + O\left(\frac{1}{y^2}\right), \tag{2.3}$$

which leads to

$$\begin{aligned} \exp(-\theta(t+y)) &= \exp\left(-\theta t + \operatorname{sgn}(t+y)\left(-\frac{\pi}{2} + \varphi\right)y\right) \\ &= \exp\left(-\theta t - \frac{\pi}{2}|y| + \sigma + O\left(\frac{1}{y^2}\right)\right) \\ &= \exp\left(-\theta t - \frac{\pi}{2}|y| + \sigma\right)\left(1 + O\left(\frac{1}{y^2}\right)\right). \end{aligned} \tag{2.4}$$

Next we easily see that

$$r^\sigma = (\sigma^2 + (t+y)^2)^{\sigma/2} = |y|^\sigma\left(1 + O\left(\frac{1}{y}\right)\right). \tag{2.5}$$

By the facts that

$$\log r = \log|y| + \frac{1}{2}\log\left(\left(1 + \frac{t}{y}\right)^2 + \frac{\sigma^2}{y^2}\right) = \log|y| + \frac{t}{y} + O\left(\frac{1}{y^2}\right)$$

and that $|\theta| = \frac{\pi}{2} - \varphi = \frac{\pi}{2} + O\left(\frac{1}{y}\right)$ which is deduced from (2.3), we compute that

$$(t+y)\log r + \theta\sigma = (t+y)\log|y| + \operatorname{sgn}(y)\frac{\pi}{2}\sigma + t + O(1). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) gives that

$$\begin{aligned} z^z &= r^\sigma e^{-\theta(t+y) + i((t+y)\log r + \theta\sigma)} \\ &= |y|^{s+iy} e^{s + \operatorname{sgn}(y)\frac{\pi i z}{2}}\left(1 + O\left(\frac{1}{y}\right)\right). \end{aligned}$$

The proof is complete. ☐

Lemma 2.3. *As $|y| \to \infty$, it holds that*

$$\Gamma(s+iy) = |y|^{z-\frac{1}{2}} \exp\left(i\left(\mathrm{sgn}(y)\frac{\pi}{2}z - (2-\mathrm{sgn}(y))\frac{\pi}{4} - y\right)\right)\sqrt{2\pi}\left(1+O\left(\frac{1}{y}\right)\right)$$

with $s \in \mathbf{C}$ being fixed and $z = s+iy$.

Proof. Stirling's formula and the previous lemma show that

$$\begin{aligned}\Gamma(z) &= e^{-z}z^{z-\frac{1}{2}}\sqrt{2\pi}\left(1+O\left(\frac{1}{|z|}\right)\right)\\ &= |y|^z e^{i(\mathrm{sgn}(y)\frac{\pi z}{2}-y)} z^{-1/2}\sqrt{2\pi}\left(1+O\left(\frac{1}{|z|}\right)\right).\end{aligned}$$

By $y = e^{\frac{1-\mathrm{sgn}(y)}{2}\pi i}|y|$ and thus

$$(s+iy)^{-1/2} = |y|^{-1/2}e^{-(2-\mathrm{sgn}(y))\pi i/4}(1+O(y^{-1})).$$

We have the conclusion. □

Lemma 2.4. *As $|m| \to \infty$,*

$$G(s, \hat{\chi}\lambda^m) = 2^{-r_1 s}|m|^{n(s-\frac{1}{2})}\prod_{v=1}^{n}\tilde{c}_v(m)|b_v|^{s-\frac{1}{2}}\left(1+O\left(\frac{1}{m}\right)\right),$$

where $\tilde{c}_v(m)$ is given by

$$\tilde{c}_v(m) = (-1)^{\frac{a_v}{2}}(1-\mathrm{sgn}(mb_v)i)\left(\frac{|mb_v|}{2e}\right)^{imb_v}$$

for $1 \le v \le r_1$, and by

$$\tilde{c}_v(m)^2 = -\mathrm{sgn}(mb_v)ic_v(m)$$

for $r_1+1 \le v \le r_1+r_2$ with

$$c_v(m) = \begin{cases} 1-(-1)^{\frac{ma_v}{2}}(2s-1)\log\frac{2ib_v}{a_v+2ib_v} & (ma_v \in 2\mathbf{Z})\\ 1-(-1)^{\frac{ma_v-1}{2}}(2s-1)\log\frac{2ib_v}{a_v+2ib_v} & (ma_v \in 1+2\mathbf{Z}).\end{cases}$$

Proof. Huxley [H] p.115 computes that

$$\begin{aligned}&G_v(s,\hat{\chi}\lambda)\\ &= \begin{cases} 2^{1-s-ib_v}\pi^{-1/2}\cos\frac{(s+ib_v)\pi}{2}\Gamma(s+ib_v) & (1\le v\le r_1,\ a_v=0)\\ 2^{1-s-ib_v}\pi^{-1/2}\sin\frac{(s+ib_v)\pi}{2}\Gamma(s+ib_v) & (1\le v\le r_1,\ a_v=1)\\ \frac{\sin(s+\frac{1}{2}|a_v|+ib_v)\pi}{\pi}\Gamma(s+\frac{|a_v|}{2}+ib_v)\Gamma(s-\frac{|a_v|}{2}-ib_v). & (r_1+1\le v\le r_1+r_2)\end{cases}\end{aligned} \tag{2.7}$$

When $1 \le v \le r_1$, the cos or sin part in (2.7) is asymptotically equal to

$$\frac{(-1)^{\frac{a_v+1}{2}}}{2} e^{\mathrm{sgn}(b_v)\frac{\pi}{2}(b_v - i(s+1))}$$

up to an exponentially small error term. Hence from Lemma 2.3 we compute that

$$G_v(s, \hat{\chi}\lambda^m) = \frac{(-1)^{\frac{a_v}{2}}}{2^{s+imb_v-\frac{1}{2}}} |mb_v|^{s+imb_v-\frac{1}{2}} e^{-(\mathrm{sgn}(mb_v)\frac{\pi}{4}+mb_v)i} \left(1 + O\left(\frac{1}{m}\right)\right).$$

Since $e^{-\mathrm{sgn}(mb_v)\frac{\pi}{4}i} = 2^{-1/2}(1 - \mathrm{sgn}(mb_v)i)$, we have

$$G_v(s, \hat{\chi}\lambda^m) = 2^{-s}\tilde{c}_v(m)|mb_v|^{s-\frac{1}{2}} \left(1 + O\left(\frac{1}{m}\right)\right). \tag{2.8}$$

When $r_1 + 1 \le v \le r_1 + r_2$, (2.7) can further be calculated as follows. We iterate the formula of the Gamma function $\Gamma(z+1) = z\Gamma(z)$ to get

$$G_v(s, \hat{\chi}\lambda)$$
$$= \begin{cases} \frac{(-1)^{\frac{|a_v|}{2}} \sin(s+ib_v)\pi}{\pi} \Gamma(s+ib_v)\Gamma(s-ib_v) \prod_{k=1}^{\frac{|a_v|}{2}} \frac{s+ib_v+k-1}{s-ib_v-k} & (a_v \in 2\mathbf{Z}) \\ \frac{(-1)^{\frac{|a_v|-1}{2}} \cos(s+ib_v)\pi}{\pi} \Gamma(s+\frac{1}{2}+ib_v)\Gamma(s-\frac{1}{2}-ib_v) \prod_{k=1}^{\frac{|a_v|-1}{2}} \frac{s+ib_v+k-\frac{1}{2}}{s-ib_v-k-\frac{1}{2}} & (a_v \in 1+2\mathbf{Z}) \end{cases} \tag{2.9}$$

We estimate the product over k in (2.9) for $G_v(s, \hat{\chi}\lambda^m)$ as $|m| \to \infty$. When $ma_v \in \mathbf{Z}$, it is

$$\begin{aligned} \prod_{k=1}^{\frac{|ma_v|}{2}} \frac{s+imb_v+k-1}{s-imb_v-k} &= \prod_{k=1}^{\frac{|ma_v|}{2}} \left(-1 + \frac{2s-1}{s-imb_v-k}\right) \\ &= (-1)^{\frac{|ma_v|}{2}} - (2s-1)\sum_{k=1}^{\frac{|ma_v|}{2}} \frac{1}{s-imb_v-k} + O\left(\frac{1}{m}\right) \\ &= (-1)^{\frac{|ma_v|}{2}} - (2s-1)\log\frac{2ib_v}{a_v+2ib_v} + O\left(\frac{1}{m}\right). \end{aligned}$$

When $ma_v \in 1+2\mathbf{Z}$, we have

$$\prod_{k=1}^{\frac{|a_v|-1}{2}} \frac{s+ib_v+k-\frac{1}{2}}{s-ib_v-k-\frac{1}{2}} = (-1)^{\frac{|ma_v|-1}{2}} - (2s-1)\log\frac{2ib_v}{a_v+2ib_v} + O\left(\frac{1}{m}\right).$$

Next by applying Lemma 2.3 we estimate the product of the gamma functions in (2.9) as $|m| \to \infty$ as follows:

$$\Gamma(s+imb_v)\Gamma(s-imb_v) = -2\pi|mb_v|^{2s-1}e^{-\pi|mb_v|}\left(1 + O\left(\frac{1}{m}\right)\right)$$

and

$$\Gamma(s+imb_v+\frac{1}{2})\Gamma(s-imb_v-\frac{1}{2}) = -2\pi i \mathrm{sgn}(mb_v)|mb_v|^{2s-1}e^{-\pi|mb_v|}\left(1+O\left(\frac{1}{m}\right)\right).$$

Then (2.9) is written as

$$G_v(s,\hat{\chi}\lambda^m) = \tilde{c}_v(m)^2 e^{-\mathrm{sgn}(mb_v)is\pi}|mb_v|^{2s-1}\left(1+O\left(\frac{1}{m}\right)\right). \tag{2.10}$$

Multiplying (2.8) and (2.10) for all v leads to the lemma. □

Corollary 2.5. *Put* $\tilde{G}(s,\hat{\chi}\bar{\lambda}^m) = |m|^{-n(s-\frac{1}{2})}G(s,\hat{\chi}\bar{\lambda}^m)$, *and we have*

$$\tilde{G}(s,\hat{\chi}\bar{\lambda}^m) = O(1)$$

as $|m| \to \infty$.

Proof. By the definition of $\tilde{c}_v(m)$ in Lemma 2.4, we have $\tilde{c}_v(m) = O(1)$ as $|m| \to \infty$. □

We follow the method of Luo-Sarnak [LS] for proving the approximate functional equation. Put for $\sigma > 1/2$

$$F(X) = \frac{1}{2\pi i}\int_{(\sigma)}\Gamma(s+l)X^{-s}\frac{ds}{s},$$

$$F(w,X) = \frac{1}{2\pi i}\int_{(\sigma)}\Gamma(-s+l)G(\bar{w}+s,\hat{\chi}\bar{\lambda}^m)X^{-s}\frac{ds}{s}$$

where $w \in \mathbf{C}$ is fixed with $\mathrm{Re}(w) = 1/2$.

Lemma 2.6. *Both* $F(X)$ *and* $F(w,X)$ *rapidly decrease as* $X \to \infty$, *and we have*

$$F(X) = O(1) \quad \textit{for} \quad X \ll 1$$

$$F(w,X) = O(1) \quad \textit{for} \quad X \ll |m|^n$$

Proof. By integrating by parts

$$F(X) = \int_X^\infty e^{-\xi}\xi^{l-1}d\xi = e^{-X}X^{l-1} + (l-1)\int_X^\infty e^{-\xi}\xi^{l-2}d\xi.$$

It is $O(1)$ as $X \to 0$, and is estimated as

$$F(X) = e^{-X}X^{l-1}\left(1+O\left(\frac{1}{X}\right)\right)$$

as $X \to \infty$.

Next we change the order of integrals to get

$$F(w,X) = \int_0^\infty e^{-\xi}\xi^{l-1}\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{|m|^n\xi}{X}\right)^{s+it_0}\left(\frac{\xi}{X}\right)^{-it_0}\tilde{G}(\bar{w}+s,\hat{\chi}\bar{\lambda}^m)\frac{ds}{s}d\xi,$$

where we put $w = \frac{1}{2} + it_0$. We first compute the inner integral on s. When $X > |m|^n\xi$, we shift the contour from (σ) to $(+\infty)$. By Lemma 2.1 there are no poles in the region. and we deduce that the integral goes to 0. When $X < |m|^n\xi$, by shifting the contour to $(-\infty)$, we have

$$\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{|m|^n\xi}{X}\right)^{s+it_0}\left(\frac{\xi}{X}\right)^{-it_0} G(\bar{w}+s,\hat{\chi}\bar{\lambda}^m)\frac{ds}{s}$$
$$= G(w,\hat{\chi}\bar{\lambda}^m) + \sum_{\rho}\left(\frac{|m|^n\xi}{X}\right)^{\rho+it_0}\frac{\mathrm{Res}_{s=\rho}\tilde{G}(\bar{w}+s,\hat{\chi}\bar{\lambda}^m)}{\rho}\left(\frac{\xi}{X}\right)^{-it_0} \tag{2.11}$$

where the sum is taken over the poles ρ of $G(\bar{w}+s,\hat{\chi}\bar{\lambda}^m)$. By Lemma 2.1 we see that the sum absolutely converges. We also have $G(w,\hat{\chi}\bar{\lambda}^m) = O(|m|^{n(w-\frac{1}{2})}) = O(1)$ since $\mathrm{Re}(w) = \frac{1}{2}$. Therefore we have

$$F(w,X) \ll \int_{\frac{X}{|m|^n}}^{\infty} e^{-\xi}\xi^{l-1}d\xi,$$

which can be estimated again by integrating by parts. □

Proposition 2.7. *Let $a_m(k)$ be the k-th coefficient in the Dirichlet series expansion of the Hecke L-function $L(w,\chi\lambda^m)$, that is,*

$$L(w,\chi\lambda^m) = \sum_{k=1}^{\infty}\frac{a_m(k)}{k^w}.$$

Then with $\mathrm{Re}(w) = 1/2$ *and w, χ, λ being fixed, it holds that*

$$L(w,\chi\lambda^m)\Gamma(l) = \sum_{k=1}^{\infty}\frac{a_m(k)}{k^w}F(kx) - W(\chi\lambda^m)A^{1-2w}\sum_{k=1}^{\infty}\frac{\overline{a_m(k)}}{k^{\bar{w}}}F(w,k/xA^2),$$

where $x > 0$ and $l \in \mathbf{Z}$, $l > 0$.

Proof. Consider the integral

$$I = \frac{1}{2\pi i}\int_{(\sigma)} L(w+s,\chi\lambda^m)\Gamma(s+l)x^{-s}\frac{ds}{s}, \tag{2.12}$$

where $\sigma > 1/2$. We compute

$$I = \sum_{k=1}^{\infty}\frac{a_m(k)}{k^w}F(kx).$$

On the other hand, since the integrand in (2.12) has a pole at $s = 0$, Cauchy's theorem shows

$$I = L(w,\chi\lambda^m)\Gamma(l) + \frac{1}{2\pi i}\int_{(-\sigma)} L(w+s,\chi\lambda^m)\Gamma(s+l)x^{-s}\frac{ds}{s}.$$

By changing the variable $s \mapsto -s$, the second term $I - L(w, \chi\lambda^m)\Gamma(l)$ is equal to

$$-\frac{1}{2\pi i}\int_{(\sigma)} L(w-s, \chi\lambda^m)\Gamma(-s+l)x^s \frac{ds}{s}.$$

We deduce by the functional equation (2.1) that

$$\begin{aligned} I - &L(w, \chi\lambda^m)\Gamma(l) \\ &= -\frac{1}{2\pi i}\int_{(\sigma)} L(\bar{w}+s, \bar{\chi}\bar{\lambda}^m)\Gamma(-s+l)W(\chi\lambda^m)A^{1-2(w-s)}G(\bar{w}+s, \hat{\chi}\bar{\lambda}^m)x^s\frac{ds}{s} \\ &= -\frac{W(\chi\lambda^m)A^{1-2w}}{2\pi i}\sum_{k=1}^{\infty}\frac{\overline{a_m(k)}}{k^{\bar{w}}}\int_{(\sigma)}\Gamma(-s+l)G(\bar{w}+s, \hat{\chi}\bar{\lambda}^m)\left(\frac{k}{A^2x}\right)^{-s}\frac{ds}{s}. \end{aligned}$$

This gives the conclusion. □

Corollary 2.8. *For fixed $w \in \mathbf{C}$ with $\mathrm{Re}(w) = 1/2$, we have*

$$L(w, \chi\lambda^m) = O\left(\sum_{\substack{\mathfrak{b}:\text{ideal} \\ N(\mathfrak{b})<|m|^{n/2}}} \frac{\chi\lambda^m(\mathfrak{b})}{N(\mathfrak{b})^w}\right)$$

Proof. Choose $x = |m|^{-n/2}$, and the corollary follows from $|W(\chi\lambda^m)| = 1$. □

3. Reduction to Cubes. Let $A, B \in \mathbf{R}$ satisfy $0 < A < B \leq 2A$. Let $L(\mathfrak{a})$ be the set of principal ideals divisible by $\mathfrak{a}$. In this section we estimate the exponential sum

$$S_{\mathfrak{a}}(A,B) = \sum_{\substack{\mathfrak{b}\subset L(\mathfrak{a}) \\ A\leq N(\mathfrak{b})<B}} \frac{\chi\lambda^m(\mathfrak{b})}{N(\mathfrak{b})^{it}} \tag{3.1}$$

as $|m| \to \infty$.

Lemma 3.1. *Let $\varepsilon_1, ..., \varepsilon_r$ $(r = r_1 + r_2 - 1)$ be the fundamental units of K. Let (α_{ik}) be the inverse of the square matrix $(\log|\varepsilon_k^{(i)}|)_{1\leq i,k\leq r}$. Let $\mathfrak{M}$ be the set of $\mu \in \mathfrak{a}\cap O_K$ such that $A \leq |N(\mu)| < B$ and that $0 \leq t_k(\mu) < 1$ where for $k = 1, ..., r$,*

$$t_k(\mu) = \sum_{i=1}^{r}\alpha_{ik}\left(\log|\mu^{(i)}| - \frac{1}{n}\log|N(\mu)|\right).$$

Then

$$S_{\mathfrak{a}}(A,B) = \frac{1}{w}\sum_{\mu\in\mathfrak{M}}\frac{\chi\lambda^m(\mu)}{N(\mu)^{it}},$$

where w is the number of roots of unity in K.

Proof. We follow the method of Mitsui [M] Lemma 1. Put $a_k = [t_k(\mu)]$ $(k = 1, ..., r)$. We have $\mu_1 := \mu\prod_{k=1}^{r}\varepsilon_k^{-a_k} \in \mathfrak{M}$. The product $\mu_1\varepsilon$ $(\varepsilon \in O_K^\times)$ belongs to $\mathfrak{M}$ if and only if ε is a root of unity. Therefore the number of generators of the ideal (μ) which belongs to $\mathfrak{M}$ is equal to w. □

Let $\gamma_1, ...\gamma_n$ be the basis of $\mathfrak{a}$. Let J be a subset of $\{1, ..., r_1\}$. We define a subset X_J of $\mathbf{R}^n$ as

$$X_J := \left\{ (x_1, ..., x_n) \;\middle|\; \begin{array}{lc} \sum_{i=1}^{n} x_i\gamma_i^{(q)} > 0 & \text{for } q \in J \\ \sum_{i=1}^{n} x_i\gamma_i^{(q)} < 0 & \text{for } q \notin J,\ 1 \leq q \leq r_1 \\ \sum_{i=1}^{n} x_i\gamma_i^{(q)} \neq 0 & \text{for } r_1 + 1 \leq q \leq n \end{array} \right\}.$$

A map $f_J : X_J \to \mathbf{R}^n$ is defined as follows:

$$f_J(x_1, ..., x_n) = (y_0, y_1, ..., y_r, \theta_1, ..., \theta_{r_2})$$

where

$$\begin{aligned} y_0 &= \sum_{q=1}^{n} \log\left|\sum_{i=1}^{n} x_i\gamma_i^{(q)}\right| \\ y_k &= \sum_{q=1}^{r} \alpha_{qk}\left(\log\left|\sum_{i=1}^{n} x_i\gamma_i^{(q)}\right| - \frac{1}{n}y_0\right) \quad (k = 1, ..., r) \\ \theta_p &= \arg\sum_{i=1}^{n} x_i\gamma_i^{(p+r_1)} \quad (p = 1, ..., r_2). \end{aligned}$$

We see that f_J is injective. Put

$$V = \left\{ (y_0, y_1, ..., y_r, \theta_1, ..., \theta_{r_2}) \;\middle|\; \begin{array}{c} \log A \leq y_0 < \log B \\ 0 \leq y_k < 1 \quad (k = 1, ..., r) \\ 0 \leq \theta_p < 2\pi \quad (p = 1, ..., r_2) \end{array} \right\}.$$

Then $\mu = \sum_{i=1}^{n} m_i\gamma_i \in \mathfrak{a} \cap O_K$ belongs to $\mathfrak{M}$ if and only if $(m_1, ..., m_n) \in f_J^{-1}(V)$ for some J. Putting for $\mu \in K^\times$

$$F(\mu) := \exp(-it\log|N(\mu)| + \log\chi(\mu) + m\log\lambda(\mu)),$$

we have

$$S_{\mathfrak{a}}(A, B) = \frac{1}{w} \sum_{J \subset \{1,2,...,r_1\}} \sum_{\substack{(m_1,...,m_n) \in f_J^{-1}(V) \\ m_j \in \mathbf{Z}}} F\left(\sum_{i=1}^{n} m_i\gamma_i\right). \tag{3.2}$$

Lemma 3.2. *For $k_1, ..., k_n \in \mathbf{Z}$, we define a cube $Q \subset \mathbf{R}^n$ as*

$$\begin{aligned} Q &= Q(k_1, ..., k_n; M) \\ &= \{(x_1, ..., x_n) \mid k_qM \leq x_q < (k_q + 1)M \quad (q = 1, ..., n)\}, \end{aligned} \tag{3.3}$$

where $M = [A^{8/9n}]$. Then

$$\sum_{\substack{(m_1,...,m_n) \in f_J^{-1}(V) \\ m_j \in \mathbf{Z}}} F\left(\sum_{i=1}^{n} m_i\gamma_i\right) = \sum_{Q \subset f_J^{-1}(V)} \sum_{(m_1,...,m_n) \in Q} F\left(\sum_{i=1}^{n} m_i\gamma_i\right) + O(A^{1-\frac{1}{9n}}).$$

Proof. This is a corollary of the proof of Mitsui's lemma ([M] Lemma 2). His lemma is stated in the case when $M = [A^{6/7n}]$ with remainder $O(A^{1-\frac{1}{7n}})$, but the proof does not rely on the exponent of A. Actually in his proof he shows in page 237 that the remainder term is bouded by $\delta A \ll MA^{1-\frac{1}{n}}$ with δ the maximum of the diameters of the $f_J(Q)$ having points with V in common. □

4. Main Theorem.

Lemma 4.1 ([T] Lemma 5.11). *Let $f(x)$ be real and have continuous derivatives up to the third order, and let $\lambda_3 \le f'''(x) \le h\lambda_3$, or $\lambda_3 \le -f'''(x) \le h\lambda_3$, and $b-a \ge 1$. Then*

$$\sum_{a<n\le b} e^{2\pi i f(n)} = O(h^{1/2}(b-a)\lambda_3^{1/6}) + O((b-a)^{1/2}\lambda_3^{-1/6}).$$

Lemma 4.2. *For fixed $k_1, ..., k_n \ge 0$ and the cube $Q = Q(k_1, ..., k_n)$ defined in Lemma 3.2, we have*

$$\sum_{(m_1,\ldots,m_n)\in Q} F\left(\sum_{i=1}^{n} m_i\gamma_i\right) \ll M^{n-\frac{1}{2}}|m|^{\frac{1}{6}} + M^n|m|^{-\frac{1}{6}}.$$

Proof. The left hand side of the lemma is

$$\sum_{m_1}\cdots\sum_{m_{n-1}}\sum_{m_n} \exp\left(-it\log|N(\mu)| + \log\chi(\mu) + m\log\left(\prod_{v=1}^{r_1+r_2}|\mu^{(v)}|^{ib_v}\prod_{v=r_1+1}^{r_1+r_2}\left(\frac{\mu^{(v)}}{|\mu^{(v)}|}\right)^{a_v}\right)\right),$$

where m_q runs through the integers in $k_qM \le m_q < (k_q+1)M$ $(q = 1, ..., n)$ with $\mu = m_1\gamma_1 + \cdots + m_n\gamma_n$. We will trivially estimate the sum over $m_1, ..., m_{n-1}$, and will obtain a nontrivial estimate for the sum over m_n by van der Corput's technique. In what follows we write $\mu^{(v)} = x_1\gamma_1^{(v)} + \cdots + x_n\gamma_n^{(v)}$ with real variables $x_1, ..., x_n$. Fix $x_1, ..., x_{n-1}$ and put

$$f(x_n) = -\frac{1}{2\pi} \times \left(t\sum_{v=1}^{n}\log|\mu^{(v)}| - \arg\chi(\mu) - m\left(\sum_{v=1}^{n} b_v\log|\mu^{(v)}| - i\sum_{v=r_1+1}^{r_1+r_2} a_v\log\frac{\mu^{(v)}}{|\mu^{(v)}|}\right)\right).$$

Let

$$A^{(v)} = \sum_{j=1}^{n-1} x_j\mathrm{Re}(\gamma_j^{(v)}) \quad\text{and}\quad B^{(v)} = \sum_{j=1}^{n-1} x_j\mathrm{Im}(\gamma_j^{(v)}).$$

Then we have

$$\mathrm{Re}(\mu^{(v)}) = \sum_{j=1}^{n} x_j\mathrm{Re}(\gamma_j^{(v)}) = x_n\mathrm{Re}(\gamma_n^{(v)}) + A^{(v)}$$

$$\mathrm{Im}(\mu^{(v)}) = \sum_{j=1}^{n} x_j\mathrm{Im}(\gamma_j^{(v)}) = x_n\mathrm{Im}(\gamma_n^{(v)}) + B^{(v)}.$$

We compute

$$-i\frac{\partial}{\partial x_n}\log\frac{\mu^{(v)}}{|\mu^{(v)}|} = \frac{\partial}{\partial x_n}\left(\tan^{-1}\frac{\mathrm{Im}(\mu^{(v)})}{\mathrm{Re}(\mu^{(v)})}\right)$$
$$= \frac{A^{(v)}\mathrm{Im}(\gamma_n^{(v)}) - B^{(v)}\mathrm{Re}(\gamma_n^{(v)})}{(x_n\mathrm{Re}(\gamma_n^{(v)}) + A^{(v)})^2 + (x_n\mathrm{Im}(\gamma_n^{(v)}) + B^{(v)})^2}.$$

Since $A^{(v)}$ and $B^{(v)}$ are linear in $x_1, ..., x_{n-1}$, the condition $(x_1, ..., x_n) \in Q$ implies that

$$\frac{\partial}{\partial x_n}\log\frac{\mu^{(v)}}{|\mu^{(v)}|} \approx M^{-1},$$

where $\approx$ means $\ll$ and $\gg$. Therefore

$$f'(x_n) = -\frac{1}{2\pi}\left(t\sum_{v=1}^{n}\frac{\gamma_n^{(v)}}{|\mu^{(v)}|} - m\left(\sum_{v=1}^{n}\frac{b_v\gamma_n^{(v)}}{\mu^{(v)}} - ia_v\sum_{v=r_1+1}^{r_1+r_2}\frac{\partial}{\partial x_n}\log\frac{\mu^{(v)}}{|\mu^{(v)}|}\right)\right)$$
$$\approx mM^{-1}$$

Similarly we compute

$$\frac{\partial^2}{\partial x_n^2}\log\frac{\mu^{(v)}}{|\mu^{(v)}|}$$
$$= \frac{-2(A^{(v)}\mathrm{Im}(\gamma_n^{(v)}) - B^{(v)}\mathrm{Re}(\gamma_n^{(v)}))(|\gamma_n^{(v)}|^2x_n + A^{(v)}\mathrm{Re}(\gamma_n^{(v)}) + B^{(v)}\mathrm{Im}(\gamma_n^{(v)}))}{((x_n\mathrm{Re}(\gamma_n^{(v)}) + A^{(v)})^2 + (x_n\mathrm{Im}(\gamma_n^{(v)}) + B^{(v)})^2)^2}$$
$$\approx M^{-2}$$

and

$$\frac{\partial^3}{\partial x_n^3}\log\frac{\mu^{(v)}}{|\mu^{(v)}|}$$
$$= \frac{-2(A^{(v)}\mathrm{Im}(\gamma_n^{(v)}) - B^{(v)}\mathrm{Re}(\gamma_n^{(v)}))}{((x_n\mathrm{Re}(\gamma_n^{(v)}) + A^{(v)})^2 + (x_n\mathrm{Im}(\gamma_n^{(v)}) + B^{(v)})^2)^3}$$
$$\times (|\gamma_n^{(v)}|^2|\mu^{(v)}|^2 - 4(|\gamma_n^{(v)}|^2x_n + A^{(v)}\mathrm{Re}(\gamma_n^{(v)}) + B^{(v)}\mathrm{Im}(\gamma_n^{(v)}))^2)$$
$$\approx M^{-3}.$$

These give

$$f''(x_n) \approx \frac{m}{M^2} \quad \text{and} \quad f'''(x_n) \approx \frac{m}{M^3}.$$

By Lemma 4.1 we have

$$\sum_{k_nM\le m_n<(k_n+1)M} e^{2\pi if(m_n)} \ll M\left(\frac{m}{M^3}\right)^{1/6} + M^{1/2}\left(\frac{m}{M^3}\right)^{-1/6}$$
$$= M^{1/2}m^{1/6} + Mm^{-1/6}.$$

By estimating the sums over m_q $(1 \le q \le n-1)$ trivially, we have

$$\sum_{(m_1,\dots,m_n)\in Q} F\left(\sum_{i=1}^{n} m_i\gamma_i\right) \ll M^{n-1}(M^{1/2}|m|^{1/6} + M|m|^{-1/6})$$
$$= M^{n-\frac{1}{2}}|m|^{\frac{1}{6}} + M^n|m|^{-\frac{1}{6}}.$$

This gives the conclusion. □

Corollary 4.3. *Put $S_{\mathfrak{a}}(A) := S_{\mathfrak{a}}(A, 2A)$ which is defined by (3.1). Then*

$$S_{\mathfrak{a}}(A) \ll A^{1-\frac{4}{9n}}|m|^{\frac{1}{6}} + A|m|^{-\frac{1}{6}}.$$

Proof. The number of cubes Q such that $Q \subset f_J^{-1}(V)$ in (3.3) is $O(AM^{-n}) = O(A^{1/9})$. The number of subsets J in (3.2) is a constant 2^{r_1}. Then from (3.2), Lemma 3.2 and Lemma 4.2 with $B = 2A$, we reach the conclusion. □

Theorem 4.4. *The Hecke L-functions for arbitrary number field K of degree n satisfies the following estimate for fixed vector $\mathbf{a}$ and as $|m| \to \infty$:*

$$L(\frac{1}{2} + it, \lambda^{m\mathbf{a}}) \ll_{t,\varepsilon} |m|^{\frac{n}{4}-\frac{1}{18}+\varepsilon}$$

Proof. Let $\mathfrak{a}_1, ..., \mathfrak{a}_h$ be the representatives of ideal classes of K. By Corollary 2.8 it suffices to estimate the sum

$$\sum_{\substack{\mathfrak{b}:\text{ideal} \\ 0<N(\mathfrak{b})\leq|m|^{n/2}}} \frac{\chi\lambda^m(\mathfrak{b})}{N(\mathfrak{b})^{\frac{1}{2}+it}} = \sum_{j=1}^{h} \frac{\chi\lambda^m(\mathfrak{a}_j)}{N(\mathfrak{a}_j)^{\frac{1}{2}+it}} \sum_{\substack{\mathfrak{b}\in L(\mathfrak{a}_j) \\ 0<N(\mathfrak{b})\leq|m|^{n/2}}} \frac{\chi\lambda^m(\mathfrak{b})}{N(\mathfrak{b})^{\frac{1}{2}+it}}.$$

The inner sum over principal ideals $\mathfrak{b} = (\mu)$ with $\mu \in O_K$ is estimated by the standard diadic decomposition

$$\sum_{k=1}^{\nu} \frac{S_{\mathfrak{a}}(2^k)}{2^{k/2}} \ll |m|^{\frac{n}{4}-\frac{1}{18}+\varepsilon}$$

with $\nu = \frac{n}{2}\log_2|m|$. □

References

[D] W. Duke, *Some problems in multidimensional analytic number theory*, Acta Arith. **52** (1989), 203-228.

[H] M.N. Huxley, *The large sieve inequality for algebraic number fields. II: Means of moments of Hecke zeta-functions*, Proc. London Math. Soc. **21** (1970), 108-128.

[LS] W. Luo and P. Sarnak, *Quantum ergodicity of eigenfunctions on $PSL_2(Z)\backslash H^2$*, Publ. I.H.E.S **81** (1995), 207-237.

[M] T. Mitsui, *On the prime ideal theorem*, J. Math. Soc. Japan **20** (1968), 233-247.

[Mo] L.J. Mordell, *On Hecke's modular functions, zeta functions and some other analytic functions in the theory of numbers*, Proc. London Math. Soc. **32** (1931), 501-556.

[N] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers* (1990), Springer.

[PS] Y. Petridis and P. Sarnak, *Quantum unique ergodicity for $SL_2(\mathcal{O})\backslash H^3$ and estimates for L-functions,*, J. Evol. Equ. 1 (2001).

[T] E.C. Titchmarsh, *The theory of the Riemann zeta function* (1986), Oxford.

Department of Mathematics, Keio University, 3 Hiyoshi, Kohoku, Yokohama 223-8522, Japan