# Multiple Sine Functions

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**Running title.** Multiple sine functions

**Abstract.** We construct a theory of multiple sine functions generalizing the usual sine function. As applications we have an expression for special values of zeta functions and we calculate gamma factors of Selberg zeta functions.

Mathematics Subject Classification 2000: Primary 11M36, 11M06

## 1 Introduction

Multiple sine functions are generalizations of the usual sine function

$$\mathcal{S}_1(z) = 2\sin(\pi z) = 2\pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{1.1}$$

The double sine function $\mathcal{S}_2(z)$ was firstly studied by Hölder [H] in 1886 from

$$\mathcal{S}_2(z) = e^z \prod_{n=1}^{\infty} \left( \left( \frac{1-\frac{z}{n}}{1+\frac{z}{n}} \right)^n e^{2z} \right). \tag{1.2}$$

Here we construct multiple sine functions $\mathcal{S}_r(z)$ for $r \geq 3$ also, and we study their basic properties containing periodicity, special values, and algebraic differential equations. (Basic results were reported in [Ku1, Ku2, Ku3]; see also [Ma] for a survey.)

For example, the triple sine function is given by

$$\begin{aligned} \mathcal{S}_3(z) &= e^{\frac{z^2}{2}} \prod_{n=1}^{\infty} \left( \left(1 - \frac{z^2}{n^2}\right)^{n^2} e^{z^2} \right) & (1.3) \\ &= \exp\left( \int_0^z \pi t^2 \cot(\pi t) dt \right). & (1.4) \end{aligned}$$

Then we have the following expression for the famous mysterious value $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$:

$$\zeta(3) = \frac{8\pi^2}{7} \log\left( \mathcal{S}_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}} \right). \tag{1.5}$$

We notice that this expression (1.5) originates from an attempt of Euler [E] in 1772.

After general study of multiple sine functions, this paper presents an application to the explicit calculation of the gamma factors of Selberg zeta functions in terms of multiple gamma functions $\Gamma_r(s)$ of Barnes [B]. In this process, normalized multiple sine functions and the differential equations satisfied by the multiple sine functions and expressions like (1.5) are used crucially. The result is as follows.

Let $M = \Gamma\backslash G/K$ be a compact locally symmetric space of rank one. We denote by $Z_M(s,\sigma)$ the Selberg zeta function with a unitary representation $\sigma$ of $\Gamma$:

$$Z_M(s,\sigma) = \prod_{p\in \mathrm{Prim}(M)} \prod_{\lambda\geq 0} \det(1-\sigma(p)N(p)^{-s-\lambda}),$$

where $\mathrm{Prim}(M)$ is the set of prime geodesics of $M$ with their norm $N(p) = \exp(\mathrm{length}(p))$ and $\lambda$ runs over a certain semi-lattice [G]. It is known that $Z_M(s,\sigma)$ has an analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function of order $\dim M$ and has the following functional equation [Se, G, W]:

$$Z_M(2\rho_0 - s,\sigma) = Z_M(s,\sigma)\exp\left(\mathrm{vol}(M)\dim(\sigma)\int_0^{s-\rho_0}\mu_M(it)dt\right) \tag{1.6}$$

with $\rho_0 > 0$ and $\mu_M(t)$ being the Plancherel measure.

We determine the gamma factor of $Z_M(s,\sigma)$ and obtain the functional equation of symmetric type:

**Theorem 1.1** *Let $M = \Gamma\backslash G/K$ be an even dimensional compact locally symmetric space of rank one. Put*

$$\Gamma_M(s,\sigma) = \det\left(\sqrt{\Delta_{M'}+\rho_0^2}+s-\rho_0\right)^{\mathrm{vol}(M)\dim(\sigma)(-1)^{\dim M/2}}$$

*where* det *means the regularized determinant, and $M' = G'/K$ is the compact dual symmetric space with $\Delta_{M'}$ its Laplacian. Then*

$$\Gamma_M(s,\sigma) = \begin{cases} (\Gamma_{2n}(s)\Gamma_{2n}(s+1))^{\mathrm{vol}(M)\dim(\sigma)(-1)^{(\dim M)/2-1}} & G = SO(1,2n) \\ \left(\prod_{k=0}^{n}\Gamma_{2n}(s+k)^{\binom{n}{k}^2}\right)^{\mathrm{vol}(M)\dim(\sigma)(-1)^{(\dim M)/2-1}} & G = SU(1,n) \\ \left(\prod_{k=0}^{2n-1}\Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}}\right)^{-\mathrm{vol}(M)\dim(\sigma)} & G = Sp(1,n) \\ (\Gamma_{16}(s)\Gamma_{16}(s+1)^{10}\Gamma_{16}(s+2)^{28}\Gamma_{16}(s+3)^{28} & \\ \qquad \times\Gamma_{16}(s+4)^{10}\Gamma_{16}(s+5))^{-\mathrm{vol}(M)\dim(\sigma)} & G = F_4 \end{cases} \tag{1.7}$$

*The completed zeta function $\hat{Z}_M(s,\sigma) = \Gamma_M(s,\sigma)Z_M(s,\sigma)$ satisfies the symmetric functional equation: $\hat{Z}_M(s,\sigma) = \hat{Z}_M(2\rho_0 - s,\sigma)$.*

**Remark 1.2** In the case of a Riemann surface ($G = SO(1,2) \cong SU(1,1)$), this result was proved by Vigneras [Vi] and Cartier-Voros [CV]. If $M$ is odd dimensional, the gamma factor is trivial (see Section 4). In the general case by using the Selberg trace formula we moreover have the following determinant expression similar to [Sa, Vo, Ko]:

$$\hat{Z}_M(s,\sigma) = e^{Q((s-\rho_0)^2)} \det(\Delta_M - s(2\rho_0 - s)),$$

where $Q$ is a polynomial with $\deg Q \leq \dim M/2$.

**Remark 1.3** This paper is an English version of a part of the following lecture note:

N. Kurokawa "Lectures on Multiple Sine Functions" (April-July, 1991, University of Tokyo, notes taken by S. Koyama, pp. 1-119).

**Acknowledgment.** The first author would like to express his hearty thanks to Christopher Deninger and Yu. I. Manin for their interests on the original lecture of 1991.

# 2 Multiple Sine Functions

In this section we introduce multiple sine functions, which will play the central role throughout this paper. We first introduce the multiple Hurwitz zeta function (see Barnes [B]). For $\omega_1, ..., \omega_r > 0$ and $z \in \mathbb{C}$, we put $\underline{\omega} = (\omega_1, ..., \omega_r)$ and

$$\zeta_r(s,z,\underline{\omega}) := \sum_{\mathbf{n}\geq 0} (\mathbf{n}\cdot\underline{\omega} + z)^{-s}, \tag{2.1}$$

where $\mathbf{n} = (n_1, ..., n_r) \geq 0$ means $n_i \geq 0$ and $n_i \in \mathbb{Z}$ for $1 \leq i \leq r$, and $\mathbf{n}\cdot\underline{\omega} = n_1\omega_1 + \cdots + n_r\omega_r$. The series (2.1) absolutely converges for $\mathrm{Re}(s) > r$. It is analytically continued to $s \in \mathbb{C}$ as a meromorphic function by the usual method (Barnes [B]) and holomorphic at $s \in \mathbb{C} - \{1, 2, ..., r\}$. We define the multiple gamma function by

$$\Gamma_r(z,\underline{\omega}) := \exp \zeta_r'(0,z,\underline{\omega}) = \exp\left( \frac{\partial}{\partial s}\zeta_r(s,z,\underline{\omega})\Big|_{s=0} \right),$$

which was originally studied by Barnes [B]. We note that $\zeta_1(s,z,\omega) = \omega^{-s}\zeta(s,\frac{z}{\omega})$ with $\zeta(s,z)$ the usual Hurwitz zeta function. Hence $\Gamma_1(z,\omega) = (2\pi)^{-\frac{1}{2}}\Gamma(\frac{z}{\omega})\omega^{\frac{z}{\omega}-\frac{1}{2}}$ by Lerch's formula. We define the $r$-ple sine functions $S_r(z,\underline{\omega})$ and $\mathcal{S}_r(z)$ by

$$S_r(z,\underline{\omega}) := \Gamma_r(z,\underline{\omega})^{-1}\Gamma_r(|\underline{\omega}| - z,\underline{\omega})^{(-1)^r} \tag{2.2}$$

with $|\underline{\omega}| = \omega_1 + \cdots + \omega_r$ and for $r \geq 2$

$$\begin{aligned} \mathcal{S}_r(z): &= \exp\left(\frac{z^{r-1}}{r-1}\right)\prod_{n=1}^{\infty}\left(P_r\left(\frac{z}{n}\right)P_r\left(-\frac{z}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}} \\ &= \exp\left(\frac{z^{r-1}}{r-1}\right)\prod_{n=-\infty}^{\infty}{}' P_r\left(\frac{z}{n}\right)^{n^{r-1}} \end{aligned} \tag{2.3}$$

with $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$. For example,

$$\mathcal{S}_2(z) = e^z \prod_{n=1}^{\infty} \left( \left( \frac{1-\frac{z}{n}}{1+\frac{z}{n}} \right)^n e^{2z} \right),$$
$$\mathcal{S}_3(z) = e^{\frac{z^2}{2}} \prod_{n=1}^{\infty} \left( \left( 1 - \frac{z^2}{n^2} \right)^{n^2} e^{z^2} \right).$$

We put

$$\mathcal{S}_1(z) = 2\pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = 2\sin(\pi z).$$

Taking $r = 1$ gives the usual sine function:

$$S_1(z, \omega) = \frac{2\pi}{\Gamma(\frac{z}{\omega})\Gamma(1-\frac{z}{\omega})} = 2\sin(\frac{\pi z}{\omega}).$$

We set

$$S_r(z) := S_r(z; (1, ..., 1))$$

for simplicity. Thus

$$\mathcal{S}_1(z) = S_1(z, 1) = S_1(z) = 2\sin(\pi z).$$

The double sine function $\mathcal{S}_2(z)$ was firstly studied by Hölder [H]. Later Shintani [Sh] used $S_2(z, (\omega_1, \omega_2))$ to construct class fields over real quadratic fields. (Unfortunately they did not name the functions.) To distinguish multiple sine functions, we call $\mathcal{S}_r(z)$ the primitive multiple sine function and $S_r(z, \underline{\omega})$ the normalized multiple sine function. The intimate relation between these two kinds of multiple sine functions is the main theme of this paper.

**Theorem 2.1** *The multiple sine function $S_r(z, \underline{\omega})$ satisfies the following identities:*

*(a) For $\underline{\omega} = (\omega_1, ..., \omega_r) \in \mathbb{R}_+^r$ put $\underline{\omega}(i) = (\omega_1, ..., \omega_{i-1}, \omega_{i+1}, ..., \omega_r) \in \mathbb{R}_+^{r-1}$, then we have*

$$S_r(z + \omega_i, \underline{\omega}) = S_r(z, \underline{\omega}) S_{r-1}(z, \underline{\omega}(i))^{-1}, \tag{2.4}$$

*where we put $S_0(z, \cdot) \equiv -1$.*

*(b) For a positive integer $N$, we have*

$$S_r(Nz, \underline{\omega}) = \prod_{0 \le k_i \le N-1} S_r\left( z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right), \tag{2.5}$$

*where the product is taken over the vectors $\mathbf{k} = (k_1, ..., k_r)$.*

*(c)*

$$\prod_{\substack{0\le k_i\le N-1\\ \mathbf{k}\neq \mathbf{0}}} S_r\left(\frac{\mathbf{k}\cdot\underline{\omega}}{N},\underline{\omega}\right)=N.$$

*(d)*

$$S_r(0,\underline{\omega})=0.$$

*(e) We have for any $c>0$ the homogeneity*

$$S_r(cz,c\underline{\omega})=S_r(z,\underline{\omega}).$$

*Proof.* Since

$$\begin{aligned}\zeta_r(s,z+\omega_i,\underline{\omega}) &= \sum_{\substack{n_1,\dots,n_r\ge 0\\ n_i\ge 1}}(n_1\omega_1+\cdots+n_r\omega_r+z)^{-s}\\ &= \zeta_r(s,z,\underline{\omega})-\zeta_{r-1}(s,z,\underline{\omega}(i)),\end{aligned}$$

$\Gamma_r(z+\omega_i,\underline{\omega})=\Gamma_r(z,\underline{\omega})\Gamma_{r-1}(z,\underline{\omega}(i))^{-1}$. Hence by $|\underline{\omega}|-(z+\omega_i)=|\underline{\omega}(i)|-z$, we have

$$\begin{aligned}S_r(z+\omega_i,\underline{\omega}) &= \Gamma_r(z+\omega_i,\underline{\omega})^{-1}\Gamma_r(|\underline{\omega}|-(z+\omega_i),\underline{\omega})^{(-1)^r}\\ &= \left(\Gamma_r(z,\underline{\omega})\Gamma_{r-1}(z,\underline{\omega}(i))^{-1}\right)^{-1}\left(\Gamma_r(|\underline{\omega}|-z,\omega)\Gamma_{r-1}(|\underline{\omega}(i)|-z,\underline{\omega}(i))\right)^{(-1)^r}\\ &= S_r(z,\underline{\omega})S_{r-1}(z,\underline{\omega}(i))^{-1},\end{aligned}$$

which leads to (a).

Next we put

$$\xi_r(s,z,\underline{\omega}):=-\zeta_r(s,z,\underline{\omega})+(-1)^r\zeta_r(s,|\underline{\omega}|-z,\underline{\omega}), \tag{2.6}$$

then

$$S_r(z,\underline{\omega})=\exp(\xi_r'(0,z,\underline{\omega})). \tag{2.7}$$

Since we need the details of the behavior of $\xi_r(s,z,\underline{\omega})$ around $s=0$, we describe the integral representation given by Riemann's method:

$$\xi_r(s,z,\underline{\omega})=-\frac{\Gamma(1-s)}{2\pi i}\int_C\varphi(t,z,\underline{\omega})(-t)^{s-1}dt,$$

where

$$\varphi(t,z,\underline{\omega})=\frac{-e^{-zt}+(-1)^r e^{(z-|\underline{\omega}|)t}}{(1-e^{-\omega_1 t})\cdots(1-e^{-\omega_r t})}$$

and $C$ is the union of $C_1 : +\infty \to +\varepsilon > 0$, $C_2 : \varepsilon e^{i\theta}$ $(0 \le \theta \le 2\pi)$ and $C_3 : +\varepsilon \to +\infty$. Thus $\xi_r(s, z, \underline{\omega})$ is meromorphic in $s \in \mathbb{C}$. Put the coefficients $a_m(z, \underline{\omega}) \in \mathbb{C}$ to be

$$\varphi(t, z, \underline{\omega}) = \sum_{m \ge -r} a_m(z, \underline{\omega}) t^m$$

around $t = 0$. We compute $\xi_r(-n, z, \underline{\omega}) = (-1)^n n! a_n(z, \underline{\omega})$ and in particular $\xi_r(0, z, \underline{\omega}) = a_0(z, \underline{\omega})$.

To prove (b) we first compute that

$$\begin{aligned} \zeta_r(s, Nz, \underline{\omega}) &= \sum_{n_i \ge 0} (n_1\omega_1 + \cdots + n_r\omega_r + Nz)^{-s} \\ &= N^{-s} \sum_{n_i \ge 0} \left( \frac{n_1\omega_1 + \cdots + n_r\omega_r}{N} + z \right)^{-s} \\ &= N^{-s} \sum_{0 \le k_i \le N-1} \zeta_r \left( s, z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right). \end{aligned}$$

Thus

$$\begin{aligned} \xi_r(s, Nz, \underline{\omega}) &= -\zeta_r(s, Nz, \underline{\omega}) + (-1)^r \zeta_r \left( s, N \left( \frac{|\underline{\omega}|}{N} - z \right), \underline{\omega} \right) \\ &= N^{-s} \left( - \sum_{0 \le k_i \le N-1} \zeta_r \left( s, z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right) \right. \\ &\qquad \left. + (-1)^r \sum_{0 \le k_i \le N-1} \zeta_r \left( s, \frac{|\underline{\omega}|}{N} - z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right) \right) \\ &= N^{-s} \sum_{0 \le k_i \le N-1} \xi_r \left( s, z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right). \end{aligned}$$

So we have

$$\begin{aligned} \xi_r'(0, Nz, \underline{\omega}) &= \sum_{0 \le k_i \le N-1} \xi_r' \left( 0, z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right) \\ &\qquad - (\log N) \sum_{0 \le k_i \le N-1} \xi_r \left( 0, z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right). \end{aligned}$$

Therefore it suffices to show $\xi_r(0, z, \underline{\omega}) = 0$. More generally we can show $\xi_r(-n, z, \underline{\omega}) = 0$ for any even integer $n \ge 0$. Indeed we see the function $\varphi(t, z, \underline{\omega})$ is an odd function in $t$.

Then (c) is deduced from

$$\frac{S_r(Nz, \underline{\omega})}{S_r(z, \underline{\omega})} = \prod_{\mathbf{k} \ne \mathbf{0}} S_r \left( z + \frac{\mathbf{k} \cdot \underline{\omega}}{N}, \underline{\omega} \right)$$

with $z=0$ substituted (see the proof of (d) below).

The assertion (d) follows from the following calculation:

$$\begin{aligned}\zeta_r(s,z,\underline{\omega}) &= z^{-s}+\sum_{\substack{n_i\geq 0\\ \mathbf{n}\neq\mathbf{0}}}(n_1\omega_1+\cdots+n_r\omega_r+z)^{-s}\\ &= z^{-s}-\frac{\Gamma(1-s)}{2\pi i}\int_C\left(\frac{1}{(1-e^{-\omega_1 t})\cdots(1-e^{-\omega_r t})}-1\right)e^{-zt}(-t)^{s-1}dt\\ &= z^{-s}+O(1)\end{aligned}$$

as $z\to 0$. Thus $\zeta_r'(s,z,\underline{\omega})=-z^{-s}\log z+O(1)$ and so $\zeta_r'(0,z,\underline{\omega})=-\log z+O(1)$, which leads to

$$\Gamma_r(z,\underline{\omega})\sim\frac{1}{c_r(\underline{\omega})z}$$

as $z\to 0$ with some constant $c_r(\underline{\omega})$. We reach the result by substituting $z=0$ to (2.2), since we have $\Gamma_r(|\underline{\omega}|,\underline{\omega})\neq 0,\infty$.

Lastly (e) follows from $\xi_r(s,cz,c\underline{\omega})=c^{-s}\xi_r(s,z,\underline{\omega})$ and $\xi_r(0,z,\underline{\omega})=0$. ∎

**Remark 2.2** The relation (c) indicates algebraicity of values at "division points". For example let $\epsilon=\frac{5+\sqrt{21}}{2}$ be the fundamental unit of $\mathbb{Q}(\sqrt{21})$ and take $r=2$, $\underline{\omega}=(1,\epsilon)$, $N=3$. Then (c) is:

$$\prod_{k_i=0,1,2}{}' S_2\left(\frac{k_1+k_2\epsilon}{3},(1,\epsilon)\right)=3.$$

In [Sh] Shintani proved a deep result on a similar product:

$$S_2\left(\frac{1}{3},(1,\epsilon)\right)S_2\left(1+\frac{\epsilon}{3},(1,\epsilon)\right)S_2\left(\frac{2+2\epsilon}{3},(1,\epsilon)\right)=\sqrt{\frac{\frac{1+\sqrt{21}}{2}-\sqrt{\frac{3+\sqrt{21}}{2}}}{2}}.$$

We will deal with values at division points such as

$$S_2\left(\frac{\omega_1}{2},(\omega_1,\omega_2)\right)=S_2\left(\frac{\omega_2}{2},(\omega_1,\omega_2)\right)=\sqrt{2}$$

in a forthcoming paper [KK].

**Remark 2.3** The above properties of the multiple sine functions $S_r(z,\underline{\omega})$ generalize the well-known formulas of the usual sine function $S_1(z,\omega)=2\sin\frac{\pi z}{\omega}$:

$$2\sin(N\theta)=\prod_{k=0}^{N-1}2\sin\left(\theta+\frac{k\pi}{N}\right)\qquad\text{and}\qquad\prod_{k=1}^{N-1}2\sin\frac{k\pi}{N}=N. \tag{2.8}$$

**Proposition 2.4** *We have an expression:*

$$S_r(z,\underline{\omega}) = e^{Q_{\underline{\omega}}(z)} z(z-|\underline{\omega}|)^{(-1)^{r-1}} \prod_{\mathbf{n}\geq\mathbf{0}}{}' P_r\left(-\frac{z}{\mathbf{n}\cdot\underline{\omega}}\right) P_r\left(\frac{z}{(\mathbf{n}+\mathbf{1})\cdot\underline{\omega}}\right)^{(-1)^{r-1}}$$

*with $Q_{\underline{\omega}}(z)$ a polynomial with $\deg Q_{\underline{\omega}} \leq r$ and $\mathbf{1} := (1,\cdots,1)$.*

*Proof.* We first compute

$$\frac{\partial^m}{\partial z^m}\zeta_r(s,z,\underline{\omega}) = (-1)^m s(s+1)\cdots(s+m-1)\sum_{\mathbf{n}\geq\mathbf{0}}\frac{1}{(z+\mathbf{n}\cdot\underline{\omega})^{s+m}}.$$

It is absolutely convergent for $\mathrm{Re}(s) > r-m$. In particular it converges at $s=0$ if $m \geq r+1$. We further compute that

$$\frac{\partial^{m+1}}{\partial z^m \partial s}\zeta_r(s,z,\underline{\omega}) = (-1)^m(ms^{m-1}+\cdots+(m-1)!)\sum_{\mathbf{n}\geq\mathbf{0}}\frac{1}{(z+\mathbf{n}\cdot\underline{\omega})^{s+m}} - (-1)^m s(s+1)\cdots(s+m-1)\sum_{\mathbf{n}\geq\mathbf{0}}\frac{\log(z+\mathbf{n}\cdot\underline{\omega})}{(z+\mathbf{n}\cdot\underline{\omega})^{s+m}}.$$

Therefore if $m \geq r+1$, we have

$$\left.\frac{\partial^{m+1}}{\partial z^m\partial s}\right|_{s=0}\zeta_r(s,z,\underline{\omega}) = (-1)^m(m-1)!\sum_{\mathbf{n}\geq\mathbf{0}}\frac{1}{(z+\mathbf{n}\cdot\underline{\omega})^m}$$

and

$$\begin{aligned}\left.\frac{\partial^{m+1}}{\partial z^m\partial s}\right|_{s=0}\zeta_r(s,|\underline{\omega}|-z,\underline{\omega}) &= (m-1)!\sum_{\mathbf{n}\geq\mathbf{0}}\frac{1}{(|\underline{\omega}|-z+\mathbf{n}\cdot\underline{\omega})^m}\\ &= (-1)^m(m-1)!\sum_{\mathbf{n}\geq\mathbf{0}}\frac{1}{(z-(\mathbf{n}+\mathbf{1})\cdot\underline{\omega})^m}.\end{aligned}$$

So we have for $m \geq r+1$,

$$\begin{aligned}\frac{d^m}{dz^m}\log S_r(z,\underline{\omega}) &= \left.\frac{\partial^{m+1}}{\partial z^m\partial s}\right|_{s=0}\xi_r(s,z,\underline{\omega})\\ &= (-1)^{m+1}(m-1)!\sum_{\mathbf{n}\geq\mathbf{0}}\left(\frac{1}{(z+\mathbf{n}\cdot\underline{\omega})^m}+\frac{(-1)^{r-1}}{(z-(\mathbf{n}+\mathbf{1})\cdot\underline{\omega})^m}\right),\end{aligned}$$

which is absolutely convergent for $z \notin \{-\mathbf{n}\cdot\underline{\omega}, (\mathbf{n}+\mathbf{1})\cdot\underline{\omega} \mid \mathbf{n}\geq\mathbf{0}\}$. ∎

Next we deduce some properties of $\mathcal{S}_r(z)$. We recall that

$$\mathcal{S}_r(z) = \exp\left(\frac{z^{r-1}}{r-1}\right) \prod_{n=-\infty}^{\infty}{}' P_r\left(\frac{z}{n}\right)^{n^{r-1}}, \qquad (r \geq 2)$$

where the product is taken over all nonzero integers $n$. We also defined

$$\mathcal{S}_1(z) := 2\pi z \prod_{n=-\infty}^{\infty}{}' P_1\left(\frac{z}{n}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = 2\sin\pi z.$$

**Theorem 2.5** *For $r \geq 2$, we have $\mathcal{S}_r(0) = 1$ and*

$$\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z) = \pi z^{r-1}\cot(\pi z).$$

*Consequently it holds that*

$$\mathcal{S}_r(z) = \exp\left(\int_0^z \pi t^{r-1}\cot(\pi t)dt\right), \tag{2.9}$$

*where the contour lies in $\mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\}$.*

*Proof.* We compute

$$\begin{aligned}
\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z) &= z^{r-2} + \sum_{n=1}^{\infty} n^{r-1}\left(\frac{1}{z-n} + \frac{(-1)^{r-1}}{z+n} + \frac{1}{n}\sum_{k=1}^{r}\left(\frac{z}{n}\right)^{k-1}(1+(-1)^{k+r-1})\right) \\
&= z^{r-2}\left(\frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}\right) \\
&= z^{r-1}\pi\cot(\pi z). \blacksquare
\end{aligned}$$

**Theorem 2.6** *For $r \geq 2$, the multiple sine function $\mathcal{S}_r(z)$ satisfies the following second order algebraic differential equation:*

$$\mathcal{S}_r''(z) = (1 - z^{1-r})\mathcal{S}_r'(z)^2\mathcal{S}_r(z)^{-1} + (r-1)z^{-1}\mathcal{S}_r(z) - \pi^2 z^{r-1}\mathcal{S}_r(z) \tag{2.10}$$

*with $\mathcal{S}_r(0) = 1$ and* $\mathcal{S}_r'(0) = \begin{cases} 1 & (r = 2) \\ 0 & (r \geq 3) \end{cases}$.

*Proof.* By the previous theorem we have

$$\begin{aligned}
\frac{d}{dz}\left(\frac{1}{\pi z^{r-1}}\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z)\right) &= -\frac{\pi}{\sin^2\pi z} \\
&= -\pi(\cot^2(\pi z) + 1) \\
&= -\pi\left(\left(\frac{1}{\pi z^{r-1}}\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z)\right)^2 + 1\right). \blacksquare
\end{aligned}$$

**Remark 2.7** We see (2.10) is analogous to Painlevé's differential equation of type III. Moreover the multiple cosine function

$$\mathcal{C}_r(z) = \prod_{\substack{n=-\infty \\ n\ :\text{odd}}}^{\infty} P_r\left(\frac{z}{(\frac{n}{2})}\right)^{(\frac{n}{2})^{r-1}} = \mathcal{S}_r(2z)^{2^{1-r}}\mathcal{S}_r(z)^{-1}$$

satisfies

$$\frac{\mathcal{C}_r'}{\mathcal{C}_r}(z) = -\pi z^{r-1}\tan(\pi z)$$

and the algebraic differential equation (2.10).

The polylogarithm function $\mathrm{Li}_k(x)$ is defined by

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^k}.$$

**Theorem 2.8** *For $r \geq 2$, the following representations hold:*

$$\mathcal{S}_r(z) = \exp\left(-\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-1}\frac{(2\pi i z)^k}{k!}\mathrm{Li}_{r-k}(e^{-2\pi i z}) + \frac{\pi i}{r}z^r + \frac{(r-1)!}{(2\pi i)^{r-1}}\zeta(r)\right), \quad (\mathrm{Im}(z) < 0) \quad (2.11)$$

$$\mathcal{S}_r(z) = \exp\left(-\frac{(r-1)!}{(-2\pi i)^{r-1}}\sum_{k=0}^{r-1}\frac{(-2\pi i z)^k}{k!}\mathrm{Li}_{r-k}(e^{2\pi i z}) - \frac{\pi i}{r}z^r + \frac{(r-1)!}{(-2\pi i)^{r-1}}\zeta(r)\right), \quad (\mathrm{Im}(z) > 0) \quad (2.12)$$

$$\begin{aligned}\mathcal{S}_r(z) = (2\sin\pi z)^{z^{r-1}}\exp\Bigg(&(-1)^{\frac{r}{2}}\frac{(r-1)!}{(2\pi)^{r-1}}\sum_{\substack{1\leq k\leq r-3 \\ k:odd}}\frac{(-1)^{\frac{k-1}{2}}(2\pi z)^k}{k!}\sum_{n=1}^{\infty}\frac{\cos(2\pi n z)}{n^{r-k}} \\ &-(-1)^{\frac{r}{2}}\frac{(r-1)!}{(2\pi)^{r-1}}\sum_{\substack{0\leq k\leq r-2 \\ k:even}}\frac{(-1)^{\frac{k}{2}}(2\pi z)^k}{k!}\sum_{n=1}^{\infty}\frac{\sin(2\pi n z)}{n^{r-k}}\Bigg),\end{aligned}$$
$$(2 \leq r \in 2\mathbb{Z},\ 0 \leq z < 1) \quad (2.13)$$

$$\mathcal{S}_r(z) = (2\sin\pi z)^{z^{r-1}} \exp\left( -(-1)^{\frac{r-1}{2}} \frac{(r-1)!}{(2\pi)^{r-1}} \sum_{\substack{0\le k\le r-3\\ k:even}} \frac{(-1)^{\frac{k}{2}}(2\pi z)^k}{k!} \sum_{n=1}^{\infty} \frac{\cos(2\pi n z)}{n^{r-k}} \right.$$
$$-(-1)^{\frac{r-1}{2}} \frac{(r-1)!}{(2\pi)^{r-1}} \sum_{\substack{1\le k\le r-2\\ k:odd}} \frac{(-1)^{\frac{k-1}{2}}(2\pi z)^k}{k!} \sum_{n=1}^{\infty} \frac{\sin(2\pi n z)}{n^{r-k}}$$
$$\left. +(-1)^{\frac{r-1}{2}} \frac{(r-1)!}{(2\pi)^{r-1}} \zeta(r) \right).$$
$$(3 \le r \in 1+2\mathbb{Z},\ 0 \le z < 1) \quad (2.14)$$

*Proof.* When $\mathrm{Im}(z) < 0$, by taking the contour $t = uz$ $(0 \le u \le 1)$ in (2.9) and taking into account that

$$\cot(\pi t) = i\frac{1+e^{-2i\pi t}}{1-e^{-2i\pi t}} = i\left(1 + 2\sum_{m=1}^{\infty} e^{-2\pi i m t}\right) \quad (\mathrm{Im}(t) < 0),$$

we see

$$\mathcal{S}_r(z) = \exp\left( i\int_0^z \pi t^{r-1} \left(1 + 2\sum_{m=1}^{\infty} e^{-2\pi i m t}\right) dt \right).$$

We reach the conclusion by calculating each term by integrating by parts:

$$\int_0^1 t^{r-1} e^{\alpha t} dt = (-1)^{r-1}(r-1)!\frac{e^{\alpha}}{\alpha^r}\left( \sum_{k=0}^{r-1} \frac{(-1)^k}{k!}\alpha^k - e^{-\alpha} \right).$$

This completes the proof of (2.11). When $\mathrm{Im}(z) > 0$, we deduce (2.12) similarly. For proving (2.13) and (2.14), it suffices to look at the logarithmic derivatives of the both sides since it is easy to see that the both sides equal 1 at $z = 0$. The direct calculation shows that the logarithmic derivatives of the right hand sides of (2.13) and (2.14) are equal to $\pi z^{r-1}\cot(\pi z)$ by trivial cancellations and the identity

$$\log(2\sin\pi z) = -\sum_{n=1}^{\infty} \frac{\cos(2\pi n z)}{n}.$$

Alternatively we can show that

$$\mathcal{S}_r(z) = (2\sin\pi z)^{z^{r-1}} \exp\left( -\frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-2} \frac{(2\pi i z)^k}{k!} \mathrm{Li}_{r-k}(e^{-2\pi i z}) \right.$$
$$\left. +\frac{\pi i}{r} z^r - \pi i z^r + \frac{\pi i}{2} z^{r-1} + \frac{(r-1)!}{(2\pi i)^{r-1}}\zeta(r) \right).$$
$$(0 \le z < 1) \quad (2.15)$$

We will look at the logarithmic derivative of (2.15), since the both sides of (2.15) equal 1 at $z=0$. By (2.9), the left hand side turns to

$$\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z) = \pi z^{r-1}\cot(\pi z). \tag{2.16}$$

We will show that the logarithmic derivative of the right hand side of (2.15) equals to (2.16). We have

$$\begin{aligned} &\frac{d}{dz}\log(\text{right hand side of (2.15)}) \\ &\quad= \frac{d}{dz}\left( z^{r-1}\log(2\sin(\pi z)) - \frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-2}\frac{(2\pi i z)^k}{k!}\mathrm{Li}_{r-k}(e^{-2\pi i z}) \right. \\ &\qquad\qquad \left. + \frac{\pi i}{r}z^r - \pi i z^r + \frac{\pi i}{2}z^{r-1}\right). \end{aligned} \tag{2.17}$$

The first term in the right hand side of (2.17) is equal to

$$(r-1)z^{r-2}\log(2\sin(\pi z)) + \pi z^{r-1}\cot(\pi z)$$

whose second term agrees to (2.16). So it suffices to show

$$\begin{aligned} &\frac{d}{dz}\left( -\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-2}\frac{(2\pi i z)^k}{k!}\mathrm{Li}_{r-k}(e^{-2\pi i z}) + \frac{\pi i}{r}z^r - \pi i z^r + \frac{\pi i}{2}z^{r-1}\right) \\ &\qquad\qquad = -(r-1)z^{r-2}\log(2\sin(\pi z)). \end{aligned} \tag{2.18}$$

By the formula

$$\mathrm{Li}_k'(x) = \frac{1}{x}\mathrm{Li}_{k-1}(x) \qquad (k \geq 2),$$

the former part in the left hand side of (2.18) is equal to

$$-\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-2}(L(k) - L(k+1)) = \frac{(r-1)!}{(2\pi i)^{r-1}}L(r-1), \tag{2.19}$$

where we put

$$L(k) = \begin{cases} \dfrac{(2\pi i)^k}{(k-1)!}z^{k-1}\mathrm{Li}_{r-k}(e^{-2\pi i z}) & (1 \leq k \leq r-1) \\ 0 & (k=0). \end{cases}$$

Then (2.19) is equal to

$$\begin{aligned} (r-1)z^{r-2}\mathrm{Li}_1(e^{-2\pi i z}) &= -(r-1)z^{r-2}\log(1-e^{-2\pi i z}) \\ &= -(r-1)z^{r-2}\log(e^{-\pi i z}(e^{\pi i z} - e^{-\pi i z})) \\ &= -(r-1)z^{r-2}\left(\log(2\sin(\pi z)) + \left(\frac{1}{2} - z\right)\pi i\right). \end{aligned}$$

The term $-(r-1)z^{r-2}(\frac{1}{2}-z)\pi i$ cancells with the latter part of the left hand side of (2.18). This completes the proof of (2.15) and we obtain (2.13) and (2.14) by taking the absolute value. ∎

**Examples 2.9** (a) For $r=2$

$$\mathcal{S}_2(z) = (2\sin \pi z)^z \exp\left(\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n z)}{n^2}\right).$$

In particular

$$\begin{aligned}
\mathcal{S}_2\left(\frac{1}{2}\right) &= 2^{\frac{1}{2}}, \\
\mathcal{S}_2\left(\frac{1}{4}\right) &= 2^{\frac{1}{8}} \exp\left(\frac{1}{2\pi}\sum_{\substack{n=1 \\ n:odd}}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\right) \\
&= 2^{\frac{1}{8}} \exp\left(\frac{1}{2\pi}L(2,\chi_{-4})\right),
\end{aligned}$$

where $\chi_{-4}$ is the nontrivial Dirichlet character mod 4. Hence

$$L(2,\chi_{-4}) = 2\pi \log\left(\mathcal{S}_2\left(\frac{1}{4}\right)2^{-\frac{1}{8}}\right).$$

(b) For $r=3$

$$\mathcal{S}_3(z) = (2\sin \pi z)^{z^2} \exp\left(\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi n z)}{n^3} + \frac{z}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n z)}{n^2} - \frac{1}{2\pi^2}\zeta(3)\right).$$

In particular

$$\begin{aligned}
\mathcal{S}_3\left(\frac{1}{2}\right) &= 2^{\frac{1}{4}} \exp\left(\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}\right) \\
&= 2^{\frac{1}{4}} \exp\left(-\frac{1}{\pi^2}\sum_{n:odd}\frac{1}{n^3}\right) \\
&= 2^{\frac{1}{4}} \exp\left(-\frac{7}{8\pi^2}\zeta(3)\right).
\end{aligned}$$

Hence

$$\zeta(3) = \frac{8\pi^2}{7}\log\left(\mathcal{S}_3\left(\frac{1}{2}\right)^{-1}2^{\frac{1}{4}}\right).$$

**Theorem 2.10** *The following identities hold:*

*(a)*

$$\mathcal{S}_r(z+1) = \frac{\mathcal{S}_r'(1)}{2\pi} \prod_{k=1}^{r} \mathcal{S}_k(z)^{\binom{r-1}{k-1}}.$$

*(b)*

$$\mathcal{S}_r(Nz) = A_r(N) \left( \mathcal{S}_r(z) \cdots \mathcal{S}_r\left(z + \frac{N-1}{N}\right)\right)^{N^{r-1}} \prod_{k=1}^{r-1} \prod_{a=1}^{N-1} \mathcal{S}_k\left(z + \frac{a}{n}\right)^{(-1)^{r-k}\binom{r-1}{k-1} a^{r-k} N^{k-1}}$$

*with*

$$A_r(N)^{-1} = \left( \mathcal{S}_r\left(\frac{1}{N}\right) \cdots \mathcal{S}_r\left(\frac{N-1}{N}\right)\right)^{N^{r-1}} \prod_{k=1}^{r-1} \prod_{a=1}^{N-1} \mathcal{S}_k\left(\frac{a}{n}\right)^{(-1)^{r-k}\binom{r-1}{k-1} a^{r-k} N^{k-1}}.$$

*Proof.* The logarithmic derivatives of the both sides of (a) coincide by Theorem 2.5, since

$$(z+1)^{r-1} = \sum_{k=1}^{r-1} \binom{r-1}{k-1} z^{k-1}.$$

Calculating the both sides (divided by $z$) at $z = 0$ leads to (a).

For proving (b) we first appeal to (2.9) to obtain

$$\frac{d}{dz} \log \mathcal{S}_r(Nz) = N\pi (Nz)^{r-1} \cot(\pi N z).$$

By the well-known formula (2.8), we see that

$$N \cot(\pi N z) = \frac{1}{\pi}\frac{d}{dz} \log \mathcal{S}_1(Nz) = \sum_{a=0}^{N-1} \frac{1}{\pi}\frac{d}{dz} \log \mathcal{S}_1\left(z + \frac{a}{n}\right) = \sum_{a=0}^{N-1} \cot \pi \left(z + \frac{a}{n}\right).$$

Therefore we have

$$\frac{d}{dz} \log \mathcal{S}_r(Nz) = N^{r-1} \pi z^{r-1} \sum_{a=0}^{N-1} \cot \pi \left(z + \frac{a}{N}\right).$$

Here we note that

$$z^{r-1} = \left(z + \frac{a}{N}\right)^{r-1} + \sum_{k=1}^{r-1} \binom{r-1}{k-1} \left(-\frac{a}{N}\right)^{r-k} \left(z + \frac{a}{N}\right)^{k-1}.$$

Thus

$$
\begin{aligned}
&\frac{d}{dz}\log\mathcal{S}_r(Nz)\\
= \quad & N^{r-1}\pi\sum_{a=0}^{N-1}\left(\left(z+\frac{a}{N}\right)^{r-1}+\sum_{k=1}^{r-1}\binom{r-1}{k-1}\left(-\frac{a}{N}\right)^{r-k}\left(z+\frac{a}{N}\right)^{k-1}\right)\cot\pi\left(z+\frac{a}{N}\right)\\
= \quad & N^{r-1}\sum_{a=0}^{N-1}\frac{d}{dz}\left(\log\mathcal{S}_r\left(z+\frac{a}{N}\right)+\sum_{k=1}^{r-1}(-1)^{r-k}\binom{r-1}{k-1}\left(\frac{a}{N}\right)^{r-k}\log\mathcal{S}_k\left(z+\frac{a}{N}\right)\right),
\end{aligned}
$$

which leads to the result. ∎

**Remark 2.11** The constant $\mathcal{S}_r'(1)$ appearing in (a) is completely determined in Lemma 3.1.

**Examples 2.12**

$$
\begin{aligned}
&\mathcal{S}_2'(1)=-2\pi\ ;\ \mathcal{S}_2(z+1)=-\mathcal{S}_2(z)\mathcal{S}_1(z),\\
&\mathcal{S}_3'(1)=-2\pi\ ;\ \mathcal{S}_3(z+1)=-\mathcal{S}_3(z)\mathcal{S}_2(z)^2\mathcal{S}_1(z),\\
&\mathcal{S}_4'(1)=-2\pi\exp(-6\zeta'(-2))\ ;\ \mathcal{S}_4(z+1)=-\exp(-6\zeta'(-2))\mathcal{S}_4(z)\mathcal{S}_3(z)^3\mathcal{S}_2(z)^3\mathcal{S}_1(z),\\
&\mathcal{S}_5'(1)=-2\pi\exp(-12\zeta'(-2))\ ;\ \mathcal{S}_5(z+1)=-\exp(-12\zeta'(-2))\mathcal{S}_5(z)\mathcal{S}_4(z)^4\mathcal{S}_3(z)^6\mathcal{S}_2(z)^4\mathcal{S}_1(z).
\end{aligned}
$$

**Lemma 2.13** *Let $c(r,k)\in\mathbb{Z}$ be defined by*

$$c(r,k)=\frac{1}{k}\sum_{l=1}^{k}(-1)^{l-1}\binom{k}{l}l^r.$$

*Then $c(r,k)$ satisfies that*

$$(-x)^{r-1}=\sum_{k=1}^{r}c(r,k)\binom{x+k-1}{k-1}$$

*for an indeterminate $x$. In particular $c(r,r)=(-1)^{r-1}(r-1)!$.*

*Proof.* Let $S(n,k)$ be the Stirling number of the second kind [A] (13.3.16), which is the coefficient in the expansion

$$x^n=\sum_{k=0}^{n}S(n,k)(x)_k,$$

where $(x)_k=x(x-1)\cdots(x-k+1)$. We will compute $e^{xt}$ in two ways. First we deduce that

$$e^{xt}=\left(1+(e^t-1)\right)^x=\sum_{k=0}^{\infty}\frac{(x)_k}{k!}(e^t-1)^k.$$

Secondly we calculate that

$$e^{xt} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S(n,k)(x)_k \right) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left( \sum_{n \geq k} \frac{S(n,k)}{n!} t^n \right) (x)_k.$$

Then we have

$$\sum_{n \geq k} \frac{S(n,k)}{n!} t^n = \frac{(e^t - 1)^k}{k!} = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} e^{tl} \binom{k}{l} = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \sum_{n=0}^{\infty} \frac{(tl)^n}{n!}.$$

Therefore

$$S(n,k) = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l^n = \frac{c(n,k)(-1)^{k-1}}{(k-1)!}.$$

Thus

$$x^n = \sum_{k=0}^{n} \frac{c(n,k)(-1)^{k-1}}{(k-1)!} (x)_k,$$

hence

$$x^{n-1} = \sum_{k=0}^{n} c(n,k) \binom{-x+k-1}{k-1}.$$

Now, the fact $c(n,n) = (-1)^{n-1}(n-1)!$ is seen from comparing the coefficients of $x^{n-1}$. ∎

We recall that

$$S_r(z) = S_r(z, (1, \cdots, 1)) = \Gamma_r(z)^{-1} \Gamma_r(r-z)^{(-1)^r}. \tag{2.20}$$

Its relation to $\mathcal{S}_r(z)$ is given by the following theorem (the constant $C_r$ will be determined in Theorem 3.5):

**Theorem 2.14** *For $r = 1, 2, 3, \ldots$, there exists a constant $C_r$ such that*

$$\mathcal{S}_r(z) = C_r \prod_{k=1}^{r} S_k(z)^{c(r,k)}. \tag{2.21}$$

*Proof.* We remark that $r = 1$ case holds with $C_1 = 1$ since

$$\mathcal{S}_1(z) = S_1(z) = 2\sin(\pi z)$$

and $c(1,1) = 1$. Hereafter we assume $r \geq 2$. We first deduce that

$$\mathcal{S}_r(z) = e^{P(z)} \prod_{k=1}^{r} S_k(z)^{c(r,k)} \tag{2.22}$$

for some polynomial $P(z)$ such that $\deg P \le r$. It suffices to show that

$$\frac{d^{r+1}}{dz^{r+1}} \log \mathcal{S}_r(z) = \frac{d^{r+1}}{dz^{r+1}} \log \left( \prod_{k=1}^{r} S_k(z)^{c(r,k)} \right). \tag{2.23}$$

The left hand side is equal to

$$\begin{aligned}
& \frac{d^{r+1}}{dz^{r+1}} \log \left( e^{\frac{z^{r-1}}{r-1}} \prod_{n=-\infty}^{\infty}{}' P_r\left(\frac{z}{n}\right)^{n^{r-1}} \right) \\
= & \frac{d^{r+1}}{dz^{r+1}} \left( \frac{z^{r-1}}{r-1} + \sum_{n=-\infty}^{\infty}{}' n^{r-1} \left( \log\left(1 - \frac{z}{n}\right) + \frac{z}{n} + \frac{1}{2}\left(\frac{z}{n}\right)^2 + \cdots + \frac{1}{r}\left(\frac{z}{n}\right)^r \right) \right) \\
= & (-1)^r r! \sum_{n=-\infty}^{\infty}{}' \frac{n^{r-1}}{(z-n)^{r+1}}.
\end{aligned} \tag{2.24}$$

Then as in the proof of Proposition 2.4, we have

$$\frac{d^{r+1}}{dz^{r+1}} \log S_r(z) = (-1)^r r! \sum_{n=0}^{\infty} {}_rH_n \left( \frac{1}{(z+n)^{r+1}} + \frac{(-1)^{r-1}}{(z-n-r)^{r+1}} \right),$$

where ${}_rH_n = \binom{n+r-1}{r-1}$. Therefore

$$\begin{aligned}
\sum_{k=1}^{r} c(r,k) \frac{d^{r+1}}{dz^{r+1}} \log S_k(z) = (-1)^r r! \Bigg( & \sum_{n=0}^{\infty} \left( \sum_{k=1}^{r} c(r,k)\ {}_kH_n \right) \frac{1}{(z+n)^{r+1}} \\
& + \sum_{n=0}^{\infty} \left( \sum_{k=1}^{r} c(r,k)\ {}_kH_n (-1)^{k-1} \right) \frac{1}{(z-n-k)^{r+1}} \Bigg).
\end{aligned}$$

The first sum over $k$ is equal to $(-n)^{r-1} = (-1)^{r-1} n^{r-1}$ by the previous lemma. In the second sum we replace $n$ by $n-k$ to get

$$\sum_{n=0}^{\infty} \left( \sum_{k=1}^{r} c(r,k)\ {}_kH_{n-k} (-1)^{k-1} \right) \frac{1}{(z-n)^{r+1}}.$$

Here the sum over $k$ is equal to $n^{r-1}$, because

$$\begin{aligned}
(-1)^{k-1}\ {}_kH_{n-k} &= (-1)^{k-1} \binom{n-1}{k-1} = (-1)^{k-1} \frac{(n-1)\cdots(n-k+1)}{(k-1)!} \\
&= \frac{(-n+k-1)\cdots(-n+1)}{(k-1)!} = \binom{-n+k-1}{k-1} = {}_kH_{-n}.
\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{r} c(r,k)\frac{d^{r+1}}{dz^{r+1}}\log S_k(z) &= (-1)^r r!\left(\sum_{n=0}^{\infty}\frac{(-1)^{r-1}n^{r-1}}{(z+n)^{r+1}}+\sum_{n=0}^{\infty}\frac{n^{r-1}}{(z-n)^{r+1}}\right)\\ &= \frac{d^{r+1}}{dz^{r+1}}\log \mathcal{S}_r(z)\end{aligned}$$

by (2.24), and we reach (2.23). Thus we obtain (2.22).

Next we prove by induction on $r$ that the polynomial $P(z)$ is a constant. It holds by (2.22) that

$$\mathcal{S}_r(z+1) = e^{P(z+1)}\prod_{k=1}^{r} S_k(z+1)^{c(r,k)}. \tag{2.25}$$

The left hand side is computed by Theorem 2.10 (a) as

$$c'\mathcal{S}_r(z)\mathcal{S}_{r-1}(z)^{r-1}\cdots\mathcal{S}_k(z)^{\binom{r-1}{k-1}}\cdots\mathcal{S}_1(z)$$

for some constant $c'$, which equals by (2.22) and by the assumption of induction for $\mathcal{S}_1(z), ..., \mathcal{S}_{r-1}(z)$

$$\begin{aligned}c''\left(e^{P(z)}S_r(z)^{c(r,r)}\cdots S_1(z)^{c(r,1)}\right)&\left(S_{r-1}(z)^{c(r-1,r-1)}\cdots S_1(z)^{c(r-1,1)}\right)^{r-1}\cdots S_1(z)^{c(1,1)}\\ &= c''e^{P(z)}S_r(z)^{a(r)}S_{r-1}(z)^{a(r-1)}\cdots S_1(z)^{a(1)}\end{aligned}$$

with

$$a(k) = \sum_{l=k}^{r}\binom{r-1}{l-1}c(l,k).$$

The right hand side of (2.25) is by (2.4) equal to

$$e^{P(z+1)}\prod_{k=1}^{r}\left(S_k(z)S_{k-1}(z)^{-1}\right)^{c(r,k)} = -e^{P(z+1)}S_r(z)^{c(r,r)}S_{r-1}(z)^{c(r,r-1)-c(r,r)}\cdots S_1(z)^{c(r,1)-c(r,2)}$$

since $c(r,r) = 1$ and $S_0(x,\cdot) = -1$. Thus we have by comparing the both sides of (2.25)

$$c''e^{P(z)}S_r(z)^{a(r)}\cdots S_1(z)^{a(1)} = -e^{P(z+1)}S_r(z)^{c(r,r)}S_{r-1}(z)^{c(r,r-1)-c(r,r)}\cdots S_1(z)^{c(r,1)-c(r,2)}.$$

So there exist $b(k) \in \mathbb{Z}$ such that

$$-c''e^{P(z)-P(z+1)} = \prod_{k=1}^{r} S_k(z)^{b(k)}. \tag{2.26}$$

We can compare the order of zeros at $z=-n$ $(n=1,2,3,...)$ of both sides of (2.26) by using the identity

$$\Gamma_k(z)^{-1} = e^{Q_k(z)} z \prod_{n=1}^{\infty} P_k\left(-\frac{z}{n}\right)^{{}_kH_n}$$

with some $Q_k(z)\in\mathbb{C}[z]$ such that $\deg Q_k \le k$, which can be proved in exactly the same way as in the proof of Proposition 2.4 (see the proof of Theorem 3.7 below).

Thus we have for $n=1,2,3,...$ that

$$\sum_{k=1}^{r} b(k)\ {}_kH_n = \sum_{k=1}^{r} b(k)\binom{n+k-1}{k-1} = 0. \tag{2.27}$$

The left hand side of (2.27) is a polynomial in $n$ whose degree is less than $r$. Therefore we have $b(k)=0$ for $k=1,2,...,r$. Thus $c''e^{P(z)-P(z+1)}=1$, and it is necessary that

$$P(z) = a + bz \tag{2.28}$$

for some constants $a$ and $b$ with $b\in 2\pi\sqrt{-1}\mathbb{Z}$.

It remains to show that $b=0$. We look at the identity

$$\mathcal{S}_r(2z) = e^{P(2z)} S_r(2z)^{c(r,r)} \cdots S_1(2z)^{c(r,1)}. \tag{2.29}$$

The left hand side of (2.29) is equal to by Theorem 2.10 (b) $(N=2)$

$$\begin{aligned}\mathcal{S}_r(2z) &= c'''\left(\mathcal{S}_r(z)\mathcal{S}_r\left(z+\frac{1}{2}\right)\right)^{2^{r-1}} \\ &\times \mathcal{S}_{r-1}\left(z+\frac{1}{2}\right)^{-\binom{r-1}{r-2}2^{r-2}} \mathcal{S}_{r-2}\left(z+\frac{1}{2}\right)^{\binom{r-1}{r-3}2^{r-3}} \cdots \mathcal{S}_1\left(z+\frac{1}{2}\right)^{(-1)^{r-1}}.\end{aligned}$$

So, by the assumption of the induction

$$\mathcal{S}_r(2z) = c''''e^{2^{r-1}\left(P(z)+P\left(z+\frac{1}{2}\right)\right)} \prod_{k=1}^{r} S_k(z)^{c(k)} S_k\left(z+\frac{1}{2}\right)^{d(k)} \tag{2.30}$$

for some $c(k), d(k)\in\mathbb{Z}$. On the other hand the right hand side of (2.29) is equal to

$$\begin{aligned} & e^{P(2z)}\left(S_r(z)S_r\left(z+\frac{1}{2}\right)^{\binom{r}{1}}\cdots S_r\left(z+\frac{r}{2}\right)^{\binom{r}{r}}\right)^{c(r,r)} \\ &\times\left(S_{r-1}(z)\cdots S_{r-1}\left(z+\frac{r-1}{2}\right)^{\binom{r-1}{r-1}}\right)^{c(r,r-1)}\cdots\left(S_1(z)S_1\left(z+\frac{1}{2}\right)\right)^{c(r,1)} \\ &= e^{P(2z)}\prod_{k=1}^{r} S_k(z)^{c'(k)} S_k\left(z+\frac{1}{2}\right)^{d'(k)} \end{aligned} \tag{2.31}$$

for some $c'(k), d'(k) \in \mathbb{Z}$, where we used the formulas (2.4) and (2.5) for $\underline{\omega} = (1, ..., 1)$:

$$S_r(z+1) = S_r(z)S_{r-1}(z)^{-1}$$

and

$$S_r(2z) = \prod_{k=0}^{r} S_r\left(z + \frac{k}{2}\right)^{\binom{r}{k}}.$$

By comparing the order of zeros at $z = -n$ and $z = -n - \frac{1}{2}$ of (2.30) and (2.31) for $n = 1, 2, 3, ...$, we have $c(k) = c'(k)$ and $d(k) = d'(k)$ for $k = 1, 2, ..., r$. Hence

$$c'''' e^{2^{r-1}\left(P(z)+P\left(z+\frac{1}{2}\right)\right)} = e^{P(2z)}.$$

Taking (2.28) into account, it follows that $c'''' e^{2^r bz + 2^r a + 2^{r-2} b} = e^{2bz+a}$ for all $z \in \mathbb{C}$. Hence $b = 0$ by $r \geq 2$. ∎

The following differential equation is crucial for later use.

**Theorem 2.15**

$$\frac{S_r'}{S_r}(z) = (-1)^{r-1}\binom{z-1}{r-1}\pi \cot(\pi z). \tag{2.32}$$

*Proof.* The logarithmic derivative of (2.21) shows

$$\frac{\mathcal{S}_r'}{\mathcal{S}_r}(z) = \sum_{k=1}^{r} c(r,k)\frac{S_k'}{S_k}(z).$$

So by inverting it holds for some $c'(r,k) \in \mathbb{Q}$ that

$$\frac{S_r'}{S_r}(z) = \sum_{k=1}^{r} c'(r,k)\frac{\mathcal{S}_k'}{\mathcal{S}_k}(z).$$

Hence by Proposition 2.5 it follows that

$$\frac{S_r'}{S_r}(z) = \left(\sum_{k=1}^{r} c'(r,k) z^{k-1}\right)\pi \cot(\pi z).$$

Thus it suffices to prove that

$$\sum_{k=1}^{r} c'(r,k) z^{k-1} = (-1)^{r-1}\binom{z-1}{r-1}. \tag{2.33}$$

By inverting (2.21) we have for some constant $C'_r$ that

$$S_r(z) = C'_r \prod_{k=1}^{r} \mathcal{S}_k(z)^{c'(r,k)}.$$

Let $N$ be the least common multiple of the denominators of $c'(r,k)$. We will compare the order of zeros of the both sides of

$$S_r(z)^N = {C'_r}^N \prod_{k=1}^{r} \mathcal{S}_k(z)^{Nc'(r,k)}$$

at $z = -m$ for $m = 1, 2, 3, \ldots.$ For the left hand side it is equal to the order of poles of $\Gamma_r(z)^N$ at $z = -m$, which is $N\ {}_rH_m$. On the other hand for the right hand side it is equal to $N \sum\limits_{k=1}^{r} c'(r,k)(-m)^{k-1}$. Hence

$$\sum_{k=1}^{r} c'(r,k)(-m)^{k-1} = {}_rH_m = \frac{(m+r-1)\cdots(m+1)}{(r-1)!}$$

for $m = 1, 2, 3, \ldots.$ Therefore as a polynomial in $x$, it holds that

$$\begin{aligned}\sum_{k=1}^{r} c'(r,k)x^{k-1} &= \frac{(-x+r-1)\cdots(-x+1)}{(r-1)!} = \frac{(-1)^{r-1}(x-1)\cdots(x-r+1)}{(r-1)!} \\ &= (-1)^{r-1}\binom{x-1}{r-1}. \blacksquare\end{aligned}$$

# 3 Calculations of Constants and Special Values

In this section we determine the constants $C_r$ for $r \geq 2$. As its application we obtain an expression of $\zeta(3)$ in terms of the triple sine function.

**Lemma 3.1**

$$\mathcal{S}'_r(1) = -2\pi \exp\left(-2 \sum_{\substack{1<l<r \\ odd}} \binom{r-1}{l-1} \zeta'(1-l)\right).$$

*Proof.* The case $r = 1$ is easily seen from $\mathcal{S}'_1(1) = -2\pi$. Suppose that $r \geq 2$. Then by the expressions (2.13) and (2.14) we have

$$\mathcal{S}'_r(1) = \begin{cases} -2\pi \exp\left(-\dfrac{(r-1)!}{(2\pi)^{r-1}} \displaystyle\sum_{\substack{1\leq k\leq r-3 \\ \text{odd}}} \dfrac{(2\pi)^k(-1)^{\frac{k-r+1}{2}}}{k!}\zeta(r-k)\right) & (r \in 2\mathbb{Z}) \\ -2\pi \exp\left(-\dfrac{(r-1)!}{(2\pi)^{r-1}} \displaystyle\sum_{\substack{2\leq k\leq r-3 \\ \text{even}}} \dfrac{(2\pi)^k(-1)^{\frac{k-r+1}{2}}}{k!}\zeta(r-k)\right) & (r \in 1+2\mathbb{Z},\ r \geq 3) \end{cases},$$

since the function $f(z) = (2\sin\pi z)^{z^{r-1}}$ satisfies $f(1) = 0$ and

$$f'(1) = \lim_{z\to 1}(2\sin\pi z)^{z^{r-1}-1}\frac{2\sin\pi z}{z-1} = -2\pi.$$

Putting $r - k = l$, we see the both cases equal

$$\begin{aligned}\mathcal{S}'_r(1) &= -2\pi\exp\left(-(r-1)!\sum_{\substack{1<l<r\\ \text{odd}}}\frac{(-1)^{\frac{l-1}{2}}}{(r-l)!(2\pi)^{l-1}}\zeta(l)\right)\\ &= -2\pi\exp\left(-(r-1)!\sum_{\substack{1<l<r\\ \text{odd}}}\frac{2}{(r-l)!(l-1)!}\zeta'(1-l)\right)\\ &= -2\pi\exp\left(-2\sum_{\substack{1<l<r\\ \text{odd}}}\binom{r-1}{l-1}\zeta'(1-l)\right),\end{aligned}$$

where we used

$$\zeta(l) = \frac{(2\pi)^{l-1}2(-1)^{\frac{l-1}{2}}}{(l-1)!}\zeta'(1-l)$$

coming from the functional equation for $\zeta(s)$. ∎

**Lemma 3.2** *Let* $a(r,k)\in\mathbb{Q}$ *satisfy*

$$\binom{X+r-2}{r-1} = \sum_{k=1}^{r-1}a(r,k)X^k.$$

*Then we have*

$$S_r(1) = \exp\left(-2\sum_{\substack{2\le k\le r-1\\ even}}a(r,k)\zeta'(-k)\right) = \exp\left(-2\sum_{\substack{3\le l\le r\\ odd}}a(r,l-1)\zeta'(1-l)\right). \quad (3.1)$$

*Proof.* Since

$$\zeta_r(s,z) = \zeta_r(s,z,(1,...,1)) = \sum_{n_1,\ldots,n_r}(n_1+n_2+\cdots+n_r+z)^{-s} = \sum_{n=0}^{\infty}{}_rH_n(n+z)^{-s},$$

we see

$$\begin{aligned}\zeta_r(s,1) &= \sum_{n=0}^{\infty}{}_rH_n(n+1)^{-s} = \sum_{n=1}^{\infty}{}_rH_{n-1}n^{-s} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{r-1}a(r,k)n^k\right)n^{-s}\\ &= \sum_{k=1}^{r-1}a(r,k)\zeta(s-k).\end{aligned}$$

Hence $\zeta_r'(s,1) = \sum_{k=1}^{r-1} a(r,k)\zeta'(s-k)$ and

$$\Gamma_r(1) = \exp(\zeta_r'(0,z))|_{z=1} = \exp\left(\sum_{k=1}^{r-1} a(r,k)\zeta'(-k)\right). \tag{3.2}$$

On the other hand

$$\zeta_r(s, r-1) = \sum_{n=0}^{\infty} {}_rH_n(n+r-1)^{-s} = \sum_{n=1}^{\infty} {}_rH_{n-r+1}n^{-s}.$$

Since ${}_rH_{n-r+1} = \frac{n(n-1)\cdots(n-r+2)}{(r-1)!} = (-1)^{r-1}\ {}_rH_{-n-1}$, we have

$$\begin{aligned} \zeta_r(s,r-1) &= (-1)^{r-1}\sum_{n=1}^{\infty} {}_rH_{-n-1}n^{-s} = (-1)^{r-1}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{r-1} a(r,k)(-n)^k\right)n^{-s} \\ &= (-1)^{r-1}\sum_{k=1}^{r-1} a(r,k)(-1)^k\zeta(s-k). \end{aligned}$$

Therefore

$$\Gamma_r(r-1) = \exp\left((-1)^{r-1}\sum_{k=1}^{r-1} a(r,k)(-1)^k\zeta'(-k)\right). \tag{3.3}$$

The lemma follows from (3.2), (3.3) and $S_r(1) = \Gamma_r(1)^{-1}\Gamma_r(r-1)^{(-1)^r}$.∎

**Remark 3.3** The number $a(r,k)$ is a shifted version of the Stirling number of the first kind $s(r,k)$ [A] (13.3.15):

$$(X)_r = \sum_{k=0}^{\infty} s(r,k)X^k.$$

**Lemma 3.4**

$$\sum_{k=l}^{r} c(r,k)a(r,l-1) = (-1)^{l-1}\binom{r-1}{l-1}.$$

*Proof.* We have

$$\begin{aligned} (-1)^{r-1}(X-1)^{r-1} &= \sum_{k=1}^{r} c(r,k)\binom{X+k-2}{k-1} \\ &= \sum_{k=1}^{r} c(r,k)\left(\sum_{l=1}^{k} a(k,l-1)X^{l-1}\right) \\ &= \sum_{l=1}^{r}\left(\sum_{k=l}^{r} c(r,k)a(k,l-1)\right)X^{l-1}. \end{aligned}$$

Comparing the coefficients of $X^{l-1}$ for the both sides leads to the result.∎

**Theorem 3.5** *The constant $C_r$ in Theorem 2.14 is given by*

$$C_r = \begin{cases} 1 & (r \in 2\mathbb{Z}) \\ e^{2\zeta'(1-r)} & (r \in 1+2\mathbb{Z},\ r \geq 3) \end{cases}.$$

*Proof.* Since $S_1(1) = 0$ and $S_1'(1) = -2\pi$, we have from (2.21) that

$$\mathcal{S}_r'(1) = -2\pi C_r \prod_{k=2}^{r} S_k(1)^{c(r,k)}.$$

We will compute

$$C_r = -\frac{\mathcal{S}_r'(1)}{2\pi} \prod_{k=2}^{r} S_k(1)^{-c(r,k)}.$$

It equals

$$\exp\left( -2 \sum_{\substack{1<l<r \\ \text{odd}}} \binom{r-1}{l-1} \zeta'(1-l) + 2 \sum_{\substack{1<l\leq r \\ \text{odd}}} \left( \sum_{k=l}^{r} c(r,k) a(r,l-1) \right) \zeta'(1-l) \right), \qquad (3.4)$$

since Lemma 3.2 gives that

$$S_k(1) = \exp\left( -2 \sum_{\substack{1<l\leq k \\ \text{odd}}} a(k,l-1)\zeta'(1-l) \right).$$

The sum over $k$ in (3.4) is computed as $\binom{r-1}{k-1}$ by Lemma 3.4 since $l$ are odd. The theorem follows.∎

**Examples 3.6** We have

$$\begin{aligned}
\mathcal{S}_1(z) &= S_1(z) \\
\mathcal{S}_2(z) &= S_2(z)^{-1} S_1(z) \\
\mathcal{S}_3(z) &= e^{2\zeta'(-2)} S_3(z)^2 S_2(z)^{-3} S_1(z) \\
\mathcal{S}_4(z) &= S_4(z)^{-6} S_3(z)^{12} S_2(z)^{-7} S_1(z)
\end{aligned}$$

and thus

$$\begin{aligned}
S_1(z) &= \mathcal{S}_1(z) \\
S_2(z) &= \mathcal{S}_2(z)^{-1} \mathcal{S}_1(z) \\
S_3(z) &= e^{-\zeta'(-2)} \mathcal{S}_3(z)^{\frac{1}{2}} \mathcal{S}_2(z)^{-\frac{3}{2}} \mathcal{S}_1(z) \\
S_4(z) &= e^{-2\zeta'(-2)} \mathcal{S}_4(z)^{-\frac{1}{6}} \mathcal{S}_3(z) \mathcal{S}_2(z)^{-\frac{11}{6}} \mathcal{S}_1(z).
\end{aligned}$$

**Theorem 3.7** *It holds that*

$$\mathcal{S}_r(z) = C_r \prod_{k=1}^{r} \Gamma_k(z)^{-c(r,k)} \left( \prod_{k=1}^{r} \Gamma_k(-z)^{-c(r,k)} \right)^{(-1)^{r-1}}.$$

*Proof.* By substituting (2.20) to (2.21), we have

$$\mathcal{S}_r(z) = C_r \prod_{k=1}^{r} \Gamma_k(z)^{-c(r,k)} \prod_{k=1}^{r} \Gamma_k(k-z)^{(-1)^k c(r,k)}. \tag{3.5}$$

The formula $\Gamma_k(k-z) = \Gamma_k(k-1-z)\Gamma_{k-1}(k-1-z)^{-1}$ gives that

$$\Gamma_k(k-z) = \prod_{j=0}^{k} \Gamma_j(-z)^{a(k,j)}$$

with $a(k,j) = (-1)^{k-j}\binom{k}{j} \in \mathbb{Z}$. We note $a(k,0) = (-1)^k$. Thus we can put $b(r,k) \in \mathbb{Z}$ so that

$$\prod_{k=1}^{r} \Gamma_k(k-z)^{(-1)^k c(r,k)} = \left( \prod_{k=1}^{r} \Gamma_k(-z)^{-b(r,k)} \right)^{(-1)^{r-1}}.$$

Therefore (3.5) becomes

$$\mathcal{S}_r(z) = C_r \prod_{k=1}^{r} \Gamma_k(z)^{-c(r,k)} \left( \prod_{k=1}^{r} \Gamma_k(-z)^{-b(r,k)} \right)^{(-1)^{r-1}}. \tag{3.6}$$

To show that $b(r,k) = c(r,k)$, we compute the order of zeros at $z = n$ $(n = 1, 2, 3, ...)$ for the both sides of (3.6). Some direct calculations show that

$$\left.\frac{\partial^{k+2}}{\partial z^{k+1}\partial s}\zeta_k(s,z,\underline{\omega})\right|_{s=0} = \frac{\partial^{k+1}}{\partial z^{k+1}} \log\left( z \prod_{\mathbf{n}\geq 0}{}' P_k\left( -\frac{z}{n_1\omega_1 + \cdots n_k\omega_k} \right) \right),$$

where $\mathbf{n} \geq 0$ means the same as in (2.1). Thus we have

$$\Gamma_k(z,\underline{\omega})^{-1} = e^{Q_k(z,\underline{\omega})} z \prod_{\mathbf{n}\geq 0}{}' P_k\left( -\frac{z}{\mathbf{n}\cdot\underline{\omega}} \right)$$

with some polynomial $Q_k$ whose degree in $z$ is not greater than $k$. When $\underline{\omega} = (1, ..., 1)$ it becomes

$$\Gamma_k(z)^{-1} = e^{Q_k(z)} z \prod_{n=1}^{\infty} P_k\left( -\frac{z}{n} \right)^{{}_kH_n}$$

with $\deg Q_k \leq k$. Hence the order of zeros at $z = n$ $(n = 1, 2, 3, ...)$ of (3.6) is

$$n^{r-1} = \sum_{k=1}^{r} (-1)^{r-1} b(r,k)\ {}_kH_n.$$

As this is valid for $n = 1, 2, 3, ...$, we deduce that $b(r,k) = c(r,k)$.∎

**Theorem 3.8** *(a)*

$$\zeta(3) = \frac{8\pi^2}{7} \log \left( \mathcal{S}_3 \left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}} \right). \tag{3.7}$$

*(b)*

$$\zeta(3) = \frac{16\pi^2}{3} \log \left( S_3 \left(\frac{1}{2}\right)^{-1} 2^{\frac{3}{8}} \right). \tag{3.8}$$

*(c)*

$$\zeta(3) = 4\pi^2 \log(S_3(1)). \tag{3.9}$$

*Proof.* The assertion (a) is already proved in Example 2.9(b). For proving (b) we take $r = 2$ in (2.21) with Theorem 3.5 and have $\mathcal{S}_2(z) = S_2(z)^{-1} S_1(z)$, from which it follows that $S_2(z) = \mathcal{S}_2(z)^{-1} \mathcal{S}_1(z)$. Putting $r = 3$ in (2.21) with Theorem 3.5 we have

$$\mathcal{S}_3(z) = e^{2\zeta'(-2)} S_3(z)^2 S_2(z)^{-3} S_1(z),$$

or $S_3(z) = e^{-\zeta'(-2)} \mathcal{S}_3(z)^{\frac{1}{2}} \mathcal{S}_2(z)^{-\frac{3}{2}} \mathcal{S}_1(z)$. Substituting $z = \frac{1}{2}$ gives

$$S_3\left(\frac{1}{2}\right) = e^{-\zeta'(-2)} \mathcal{S}_3 \left(\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{1}{4}},$$

where we used $\mathcal{S}_2(\frac{1}{2}) = \sqrt{2}$ in Example 2.9(a), which follows from

$$\mathcal{S}_2(z) = (2 \sin \pi z)^z \exp \left( \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n z)}{n^2} \right)$$

for $0 \leq z < 1$. So using $-\zeta'(-2) = \frac{1}{4\pi^2} \zeta(3)$ we have (3.8) from (3.7). Finally (c) follows from (3.1) for $r = 3$, which turns to $S_3(1) = \exp(-\zeta'(-2)) = \exp(\frac{1}{4\pi^2}\zeta(3))$.∎

**Expectation 3.9** We expect $\mathcal{S}_r(\mathbb{Q}) \subset \overline{\mathbb{Q}} \cup \{\infty\}$ and $S_r(\mathbb{Q}) \subset \overline{\mathbb{Q}} \cup \{\infty\}$. These would imply the transcendency of $\dfrac{\zeta(3)}{\pi^2}$ and $\zeta(3)$ by (3.7), (3.8) or (3.9) (cf. [KK] and [KW]). Here we notice only that $\mathcal{S}_2(\frac{1}{2}) = \sqrt{2}$ as in Example 2.9(a) and $S_2(\frac{1}{2}) = \sqrt{2}$ by $S_2(\frac{1}{2}) = \mathcal{S}_2(\frac{1}{2})^{-1} \mathcal{S}_1(\frac{1}{2}) = \sqrt{2}$. Similarly $\mathcal{S}_2(\frac{m}{2}) = (-1)^{[\frac{m+1}{4}]} 2^{\frac{m}{2}}$ and $S_2(\frac{m}{2}) = (-1)^{[\frac{m}{4}]} 2^{1-\frac{m}{2}}$ for odd integers $m$: the former follows from $\mathcal{S}_2(\frac{1}{2}) = \sqrt{2}$ using $\mathcal{S}_2(z+1) = -\mathcal{S}_2(z)\mathcal{S}_1(z)$ listed in Examples 2.12, and the latter is obtained by $S_2(\frac{m}{2}) = \mathcal{S}_2(\frac{m}{2})^{-1} \mathcal{S}_1(\frac{m}{2})$. (See Remark 2.2 also.)

# 4 Plancherel Measures

The Plancherel measure $\mu_M(t)$ and the constant $\rho_0 > 0$ in the functional equation (1.6) are calculated by Miatello[Mi] as follows:

(0) $G = SO(1, 2n-1)$ ($\Leftrightarrow \dim M$ : odd)

$$\begin{aligned}&\rho_0 = n-1,\\ &\mu_M(it) : \text{polynomial.}\end{aligned}$$

(1) $G = SO(1, 2n)$

$$\begin{aligned}&\dim M = 2n,\\ &\rho_0 = n - \frac{1}{2},\\ &\mu_M(it) = (-1)^n P_M(t)\pi\tan(\pi t),\\ &P_M(t) = \frac{2}{(2n-1)!} t \prod_{k=1}^{n-1}\left(t^2 - \left(k-\frac{1}{2}\right)^2\right).\end{aligned}$$

(2) $G = SU(1, 2n-1)$

$$\begin{aligned}&\dim M = 4n-2,\\ &\rho_0 = n - \frac{1}{2},\\ &\mu_M(it) = -P_M(t)\pi\tan(\pi t),\\ &P_M(t) = \frac{2}{(2n-1)!(2n-2)!} t \prod_{k=1}^{n-1}\left(t^2 - \left(k-\frac{1}{2}\right)^2\right).\end{aligned}$$

(3) $G = SU(1, 2n)$

$$\begin{aligned}&\dim M = 4n,\\ &\rho_0 = n,\\ &\mu_M(it) = -P_M(t)\pi\cot(\pi t),\\ &P_M(t) = \frac{2}{(2n)!(2n-1)!} t^3 \prod_{k=1}^{n-1}(t^2 - k^2)^2.\end{aligned}$$

(4) $G = Sp(1,n)$

$$\rho_0 = n + \frac{1}{2},$$
$$\dim M = 4n,$$
$$\mu_M(it) = P_M(t)\pi\tan(\pi t),$$
$$P_M(t) = \frac{2}{(2n+1)!(2n-1)!} t\left(t^2 - \left(n-\frac{1}{2}\right)^2\right)\prod_{k=1}^{n-1}\left(t^2 - \left(k-\frac{1}{2}\right)^2\right)^2.$$

(5) $G = F_4$

$$\rho_0 = \frac{11}{2},$$
$$\dim M = 16,$$
$$\mu_M(it) = P_M(t)\pi\tan(\pi t),$$
$$P_M(t) = \frac{2}{11!4\cdot5\cdot6\cdot7} t\left(t^2-\frac{1}{4}\right)^2\left(t^2-\frac{9}{4}\right)^2\left(t^2-\frac{25}{4}\right)\left(t^2-\frac{49}{4}\right)\left(t^2-\frac{81}{4}\right).$$

In this section we will give a new expression of the Plancherel measures, which suggests the Betti type interpretation for the coefficients. In what follows we omit the case (0) since the gamma factor is "trivial" corresponding to the nonexistence of discrete series. We use the following combinatorial results:

**Lemma 4.1** *For integers $n$ and $m$ we have:*

$$ {}_{2n}H_m + {}_{2n}H_{m-1} = \frac{(2m+2n-1)(m+1)\cdots(m+2n-2)}{(2n-1)!} \tag{4.1}$$
$$ = \mathrm{mult}(m(m+n), \Delta_{S^{2n}}), \tag{4.2}$$

$$ \sum_{k=0}^{n}\binom{n}{k}^2 {}_{2n}H_{m-k} = \frac{(2m+n)(m+1)^2\cdots(m+n-1)^2}{n!(n-1)!} \tag{4.3}$$
$$ = \mathrm{mult}(m(m+n), \Delta_{\mathbf{P}^n_{\mathbb{C}}}), \tag{4.4}$$

$$ \sum_{k=0}^{2n-1}\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1} {}_{4n}H_{m-k}$$
$$ = \frac{(2m+2n+1)(m+1)((m+2)\cdots(m+2n-1))^2(m+2n)}{(2n+1)!(2n-1)!} \tag{4.5}$$
$$ = \mathrm{mult}(m(m+2n+1), \Delta_{\mathbf{P}^n_{\mathbf{H}}}), \tag{4.6}$$

$$
\begin{aligned}
{}_{16}H_m &+ 10\ {}_{16}H_{m-1} + 28\ {}_{16}H_{m-2} + 28\ {}_{16}H_{m-3} + 10\ {}_{16}H_{m-4} + {}_{16}H_{m-5} \\
&= \tfrac{(2m+1)(m+1)(m+2)(m+3)(m+4)^2(m+5)^2(m+6)^2(m+7)^2(m+8)(m+9)(m+10)}{11!\cdot 4\cdot 5\cdot 6\cdot 7} \qquad (4.7)\\
&= \mathrm{mult}(m(m+11), \Delta_{\mathbf{P}^2_{\mathbf{O}}}). \qquad (4.8)
\end{aligned}
$$

*Proof.* The identities (4.2), (4.4) are due to Cartan [C]. More generally the results of Cahn-Wolf [CW] give (4.2), (4.4), (4.6), (4.8). These are considered as real analytic analogs of the "Hirzebruch proportionality principle".

It is easy to see (4.1). We compute (4.3) as follows:

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}^2 {}_{2n}H_{m-k} &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{m-1+2n-k}{2n-1} \\
&= \sum_{k\geq 0} \binom{n}{k}^2 \binom{m-1+n+k}{2n-1} \\
&= \sum_{k\geq 0} \binom{n}{k}^2 \sum_{j\geq 0} \binom{k}{j}\binom{m-1+n}{2n-1-j},
\end{aligned}
$$

where we used the Vandermond convolution $\sum_{k\geq 0}\binom{m}{k}\binom{n}{l-k} = \binom{m+n}{l}$. By changing the order of the sums it equals

$$
\begin{aligned}
&\sum_{j\geq 0} \binom{m-1+n}{2n-1-j} \sum_{k\geq 0} \binom{n}{k}^2 \binom{k}{j} \\
&= \sum_{j\geq 0} \binom{m-1+n}{2n-1-j}\binom{n}{j} \sum_{k\geq 0} \binom{n}{k}\binom{n-j}{n-k} \\
&= \sum_{j\geq 0} \binom{m-1+n}{2n-1-j}\binom{n}{j}\binom{2n-j}{n} \\
&= \sum_{j\geq 0} \left( \frac{2n}{m}\binom{m-1+n}{m-1}\binom{m}{n-j}\binom{n}{j} - \frac{n}{m}\binom{m-1+n}{m-1}\binom{m}{n-j}\binom{n-1}{j-1} \right) \\
&= \frac{2n}{m}\binom{m-1+n}{m-1}\binom{m+n}{n} - \frac{n}{m}\binom{m-1+n}{m-1}\binom{m+n-1}{n-1} \\
&= \frac{2m+n}{n}\binom{m-1+n}{n-1}^2.
\end{aligned}
$$

Here we reached the right hand side of (4.3).

The identity (4.5) is proved as follows:

$$\begin{aligned}
\sum_{k=0}^{2n-1} \frac{1}{2n} \binom{2n}{k}\binom{2n}{k+1}\ {}_{4n}H_{m-k}
&= \sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}\binom{4n+m-k-1}{4n-1} \\
&= \sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k+1}\binom{2n}{k}\binom{2n+m+k}{4n-1} \qquad (k \mapsto 2n-1-k) \\
&= \frac{1}{2n}\sum_{k\geq 0}\binom{2n}{k+1}\binom{2n}{k}\sum_{j\geq 0}\binom{k}{j}\binom{2n+m}{4n-1-j} \\
&= \frac{1}{2n}\sum_{j\geq 0}\binom{2n+m}{4n-1-j}\sum_{k\geq 0}\binom{2n}{k+1}\binom{2n}{k}\binom{k}{j} \\
&= \frac{1}{2n}\sum_{j\geq 0}\binom{2n+m}{4n-1-j}\sum_{k\geq 0}\binom{2n}{k+1}\binom{2n}{j}\binom{2n-j}{2n-k} \\
&= \frac{1}{2n}\sum_{j\geq 0}\binom{2n+m}{4n-1-j}\binom{2n}{j}\binom{4n-j}{2n+1} \\
&= \frac{1}{2n}\sum_{j\geq 0}\left(\frac{4n}{m}\binom{2n+m}{2n+1}\binom{m}{2n-1-j}\binom{2n}{j}\right. \\
&\qquad\qquad \left. -\frac{2n}{m}\binom{2n+m}{2n+1}\binom{m}{2n-1-j}\binom{2n-1}{j-1}\right) \\
&= \frac{2}{m}\binom{2n+m}{2n+1}\binom{m+2n}{2n-1} - \frac{1}{m}\binom{2n+m}{2n+1}\binom{m+2n-1}{2n-2}.
\end{aligned}$$

This is equal to the right hand side of (4.5).

We can verify (4.7) by direct calculations.∎

**Theorem 4.2**

$$P_M(t+\rho_0) = \begin{cases}
{}_{2n}H_t + {}_{2n}H_t & G = SO(1,2n) \\
\sum_{k=0}^{n} \binom{n}{k}^2\ {}_{2n}H_{t-k} & G = SU(1,n) \\
\sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}\ {}_{4n}H_{t-k} & G = Sp(1,n) \\
{}_{16}H_t + 10\ {}_{16}H_{t-1} + 28\ {}_{16}H_{t-2} & \\
\qquad +28\ {}_{16}H_{t-3} + 10\ {}_{16}H_{t-4} + {}_{16}H_{t-5} & G = F_4.
\end{cases}$$

*Proof.* Since we see $P_M(m+\rho_0) = \mathrm{mult}(m(m+2\rho_0), \Delta_{M'})$, the theorem holds as polynomials in $t$. ∎

The following result represents that Plancherel measures are sums of logarithmic derivatives of multiple sine functions.

**Theorem 4.3**

$$\exp\left(\int_0^{s-\rho_0} \mu_M(it)dt\right)^{(-1)^{(\dim M)/2}}$$
$$= \begin{cases} S_{2n}(s)S_{2n}(s+1) & G = SO(1,2n) \\ \prod_{k=0}^{n} S_{2n}(s+k)^{\binom{n}{k}^2} & G = SU(1,n) \\ \prod_{k=0}^{2n-1} S_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}} & G = Sp(1,n) \\ S_{16}(s)S_{16}(s+1)^{10}S_{16}(s+2)^{28}S_{16}(s+3)^{28}S_{16}(s+4)^{10}S_{16}(s+5) & G = F_4. \end{cases} \tag{4.9}$$

*Proof.* We first prove the case of $SO(1,2n)$. When $s = \rho_0 = n - \frac{1}{2}$, the left hand side clearly equals to 1. The right hand side is computed as

$$S_{2n}\left(n - \frac{1}{2}\right) S_{2n}\left(n + \frac{1}{2}\right) = \frac{\Gamma_{2n}\left(n+\frac{1}{2}\right)\Gamma_{2n}\left(n-\frac{1}{2}\right)}{\Gamma_{2n}\left(n-\frac{1}{2}\right)\Gamma_{2n}\left(n+\frac{1}{2}\right)} = 1.$$

So it suffices to compare the logarithmic derivative for the both sides of (4.9). For the right hand side we have

$$\frac{S'_{2n}}{S_{2n}}(s) + \frac{S'_{2n}}{S_{2n}}(s+1) = -\left(\binom{s-1}{2n-1} + \binom{s}{2n-1}\right)\pi\cot(\pi s).$$

On the other hand the logarithmic derivative of the left hand side of (4.9) is equal to $-P_M(s-n+\frac{1}{2})\pi\cot(\pi s)$. Therefore all we have to prove is that $P_M(s-n+\frac{1}{2}) = \binom{s-1}{2n-1} + \binom{s}{2n-1}$. We compute

$$\begin{aligned} P_M\left(s-n+\frac{1}{2}\right) &= \frac{2}{(2n-1)!}\left(s-n+\frac{1}{2}\right)\prod_{k=1}^{n-1}\left(\left(s-n+\frac{1}{2}\right)^2 - \left(k-\frac{1}{2}\right)^2\right) \\ &= (2s-2n+1)\frac{(s-1)(s-2)\cdots(s-2n+2)}{(2n-1)!} \\ &= \frac{s(s-1)\cdots(s-2n+2)}{(2n-1)!} + \frac{(s-1)(s-2)\cdots(s-2n+1)}{(2n-1)!} \\ &= \binom{s}{2n-1} + \binom{s-1}{2n-1} \end{aligned}$$

| $G$ | $K$ | $G'$ | $M'$ |
|---|---|---|---|
| $SO(1,n)$ | $SO(n)$ | $SO(1+n)$ | $S^n$ |
| $SU(1,n)$ | $SU(n)$ | $SU(1+n)$ | $\mathbf{P}^n_{\mathbb{C}}$ |
| $Sp(1,n)$ | $Sp(n)$ | $Sp(1+n)$ | $\mathbf{P}^n_{\mathbf{H}}$ |
| $F_4$ | Spin(9) | $F_4'$ | $\mathbf{P}^2_{\mathbf{O}}$ |

Table 1: Compact Duals

as desired. The other cases are similarly proved by our using Lemma 4.1 and Theorem 4.2.∎

Let $M' = G'/K$ be the compact dual symmetric spaces which are given in Table 1.

We put

$$\zeta\left(s, z, \sqrt{\Delta_{M'} + \rho_0^2}\right) := \sum_{\lambda} (\lambda + z)^{-s} \tag{4.10}$$

where the sum is taken over all eigenvalues $\lambda$ of $\sqrt{\Delta_{M'} + \rho_0^2}$ with $\Delta_{M'}$ being the Laplacian on $M'$.

**Theorem 4.4**

$$\zeta\left(s, z-\rho_0, \sqrt{\Delta_{M'} + \rho_0^2}\right)$$
$$= \begin{cases} \zeta_{2n}(s,z) + \zeta_{2n}(s,z+1) & G = SO(1,2n) \\ \displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \zeta_{2n}(s,z+k) & G = SU(1,n) \\ \displaystyle\sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}\zeta_{4n}(s,z+k) & G = Sp(1,n) \\ \zeta_{16}(s,z) + 10\zeta_{16}(s,z+1) + 28\zeta_{16}(s,z+2) & \\ \qquad +28\zeta_{16}(s,z+3) + 10\zeta_{16}(s,z+4) + \zeta_{16}(s,z+5) & G = F_4. \end{cases}$$

*Proof.* By expressing $\lambda$ in terms of an eigenvalue $\mu$ of $\Delta_{M'}$, we have

$$\zeta\left(s, z-\rho_0, \sqrt{\Delta_{M'} + \rho_0^2}\right) = \sum_{\mu} \left(\sqrt{\mu + \rho_0^2} + (z - \rho_0)\right)^{-s}.$$

Now we carry out an explicit calculation for the case $G = SU(1,n)$ by using Lemma 4.1. All other cases can be treated similarly. Since $\mu = m(m + 2\rho_0)$ for $m = 0, 1, 2, ...$, it holds that

$$\zeta\left(s, z-\rho_0, \sqrt{\Delta_{M'} + \rho_0^2}\right) = \sum_{m=0}^{\infty} \mathrm{mult}(m(m+2\rho_0), \Delta_{M'})(m+z)^{-s}.$$

Since we have $\rho_0 = \frac{n}{2}$ and $M' = \mathbf{P}^n_{\mathbb{C}}$,

$$\begin{aligned}\sum_{m=0}^{\infty} \mathrm{mult}(m(m+2\rho_0), \Delta_{\mathbf{P}^n_{\mathbb{C}}})(m+z)^{-s} &= \sum_{m=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 {}_{2n}H_{m-k}(m+z)^{-s} \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{m=0}^{\infty} {}_{2n}H_m(m+k+z)^{-s} \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 \zeta_{2n}(s, z+k). \blacksquare\end{aligned}$$

Thus in particular we see that (4.10) is regular at $s = 0$.

Let $A$ be an operator whose eigenvalues are $0 < a_1 \le a_2 \le a_3 \le \cdots$. We define the regularized determinant by

$$\det(A) = \prod_{n=1}^{\infty} a_n := \exp\left(-\zeta_A'(0)\right),$$

when the spectral zeta function $\zeta_A(s) := \sum\limits_{n=1}^{\infty} a_n^{-s}$ is regular at $s = 0$. (Cf. Deninger [D] and Manin [Ma].) For example, the multiple gamma functions have the following determinant expressions:

$$\Gamma_r(z, \underline{\omega}) = \det(D_{\underline{\omega}} + z)^{-1}$$

and

$$\Gamma_r(z) = \det(D_r + z)^{-1},$$

where

$$D_{\underline{\omega}} = \omega_1 \frac{\partial}{\partial t_1} + \cdots + \omega_r \frac{\partial}{\partial t_r} : \mathbb{C}[t_1, \cdots, t_r] \longrightarrow \mathbb{C}[t_1, \cdots, t_r]$$

and $D_r := \frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_r}$. Consequently the multiple sine functions can also be expressed by some regularized determinants.

The regularity of (4.10) at $s = 0$ allows us to define

$$\det\left(\sqrt{\Delta_{M'} + \rho_0^2} + z\right) = \exp\left(-\zeta'\left(0, z, \sqrt{\Delta_{M'} + \rho_0^2}\right)\right).$$

**Corollary 4.5**

$$\det\left(\sqrt{\Delta_{M'}+\rho_0^2}+s-\rho_0\right)^{-1}$$
$$=\begin{cases}\Gamma_{2n}(s)\Gamma_{2n}(s+1) & G=SO(1,2n)\\ \prod_{k=0}^{n}\Gamma_{2n}(s+k)^{\binom{n}{k}^2} & G=SU(1,n)\\ \prod_{k=0}^{2n-1}\Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}} & G=Sp(1,n)\\ \Gamma_{16}(s)\Gamma_{16}(s+1)^{10}\Gamma_{16}(s+2)^{28}\Gamma_{16}(s+3)^{28} & \\ \qquad\times\Gamma_{16}(s+4)^{10}\Gamma_{16}(s+5) & G=F_4\end{cases} \tag{4.11}$$

**Corollary 4.6**

$$\exp\left(\int_0^{s-\rho_0}\mu_M(it)dt\right)^{(-1)^{(\dim M)/2}}=\left(\frac{\det\left(\sqrt{\Delta_{M'}+\rho_0^2}+(s-\rho_0)\right)}{\det\left(\sqrt{\Delta_{M'}+\rho_0^2}-(s-\rho_0)\right)}\right)^{(-1)^{\dim M/2}}. \tag{4.12}$$

*Proof.* This is an immediate consequence from (4.9) and (4.11).∎

*Proof of Theorem 1.1* The identities (1.7) are obtained from (4.11). The symmetric functional equation is deduced by (4.12).∎

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