The First Eigenvalue Problem and Tensor Products of Zeta Functions

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Abstract

We obtain a new bound for the first eigenvalue of the Laplacian for Bianchi manifolds by the method of Luo, Rudnick and Sarnak. We use a recent result of Kim on symmetric power L-functions. The key idea is to take tensor products of zeta functions, and we report on our recent developments on Kurokawa's multiple zeta functions.

1 Introduction

Let Γ be a congruence subgroup of $PSL(2,\mathbb{Z})$. It acts on the upper half plane H^2 and the quotient space $M = \Gamma\backslash H^2$ is a hyperbolic noncompact orbifold. Let Δ be the hyperbolic Laplacian which operates on $L^2(M)$. Since M is not compact, in the spectra of Δ there always exist continuous ones. For general noncompact hyperbolic surfaces it is not known if discrete spectra exist. It is believed today that there are very few discrete spectra in general. However, when Γ is an arithmetic group such as a congruence subgroup of $PSL(2,\mathbb{Z})$, it is proved that the contribution from the continuous spectra is smaller than that from the discrete ones, and that the number of eigenvalues λ of Δ satisfies the Weyl law:

$$\sum_{0<\lambda\le X} 1 \sim \frac{\mathrm{vol}(M)}{4\pi} X$$

as $X \to \infty$. The question of the smallest nonzero eigenvalue λ_1 for congruence surfaces is one of the major problems. The Selberg Conjecture asserts that

$$\lambda_1 \ge \frac{1}{4}. \tag{1.1}$$

Indeed Selberg proved that $\lambda_1 \ge 3/16 = 0.185\cdots$. The first major improvement was done by Luo, Rudnick and Sarnak [LRS1], in 1995, who obtained that $\lambda_1 \ge 21/100$. Later Kim and Shahidi [KS] deduced that $\lambda_1 \ge 66/289 = 0.228\cdots$, and Kim [K] improved it to $\lambda_1 \ge 40/169 = 0.236\cdots$. The current best bound was given by Kim and Sarnak [KSa], who proved $\lambda_1 \ge 975/4096 = 0.238\cdots$.

The conjecture (1.1) is equivalent to the Riemann Hypothesis for the Selberg zeta function $Z_\Gamma(s)$ of Γ, since the nontrivial zeros of $Z_\Gamma(s)$ which correspond to an eigenvalue λ are located at $s = \frac{1}{2} \pm ir$ with $\lambda = \frac{1}{4} + r^2$.

The conjecture (1.1) is also interpreted as the infinite part of the Ramanujan Conjecture in the following manner. Let φ be a Hecke eigen Maass cusp form with the eigenvalue λ. The automorphic L-function of φ is defined by

(1.2)

$$\begin{aligned} L(s,\varphi) &= \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1} \\ &= \prod_p \det\left(I_2 - p^{-s}\begin{pmatrix}\alpha_p & 0 \\ 0 & \beta_p\end{pmatrix}\right)^{-1}, \end{aligned}$$

where the product is taken over prime numbers, $a(p)$ is the p-th Fourier coefficient of φ, and the complex numbers α_p and β_p are determined by $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = 1$. The Euler product (1.2) converges for $\mathrm{Re}(s) > 1$, and has a symmetric type functional equation. The gamma factor of $L(s,\varphi)$ is given by

$$\Gamma(s,\varphi) = \Gamma_{\mathbb{R}}(s+ir)\Gamma_{\mathbb{R}}(s-ir), \tag{1.3}$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. The Ramanujan Conjecture asserts the unitarity of the matrix $\begin{pmatrix}\alpha_p & 0 \\ 0 & \beta_p\end{pmatrix}$, which means $|\alpha_p| = 1$, and is equivalent to $\mathrm{Re}(\log_p \alpha_p) = 0$. This is analogous to the Selberg Conjecture $\lambda \ge \frac{1}{4}$, which is expressed by $\mathrm{Re}(\pm ir) = 0$. Here $\log_p \alpha_p$ and $\pm ir$ are called the Satake parameters for the automorphic form φ.

There is another important interpretation for the conjecture (1.1). We see from the form of the gamma factor (1.3) that $L(s,\varphi)$ has trivial zeros at $s=\pm ir$. If we express the conjecture (1.1) by $\mathrm{Re}(\pm ir)=0$, it is regarded as the Riemann Hypothesis for trivial zeros. Actually Jacquet and Shalika proved that for any cusp form π on $GL_n(\mathbf{A}_k)$ with k a number field, the Satake parameters μ_j $(j=1,...,n)$ for the gamma factor $\Gamma(s,\pi)$ of the automorphic L-function $L(s,\pi)$ satisfy that

$$|\mathrm{Re}(\mu_j)|<\frac{1}{2}. \tag{1.4}$$

If we apply it to the case $K=\mathbb{Q}$, $n=2$, $\pi=\varphi$, $\mu_1=ir$ and $\mu_2=-ir$, the theorem (1.4) shows $|\mathrm{Re}(\pm ir)|<\frac{1}{2}$, which can be regarded as the critical strip, giving the "trivial zero-free region" for the trivial zeros $s=\pm ir$.

Turning our eyes to three dimensional cases, the problem stands analogously. Let $\Gamma_d=PSL(2,O_d)$ be a Bianchi group, where O_d is the integer ring of the imaginary quadratic field K_d with discriminant $d<0$. The Bianchi group acts on the three dimensional hyperbolic space H^3 in the usual way, and the quotient space $M=M_d=\Gamma_d\backslash H^3$ is a noncompact arithmetic hyperbolic orbifold called the Bianchi manifold. The hyperbolic Laplacian operates on $L^2(M_d)$ and the spectral situation is the same as the two dimensional case mentioned above. The Selberg conjecture tells that the first nonzero eigenvalue λ_1 of the Laplacian should satisfy that

$$\lambda_1\geq 1. \tag{1.5}$$

The first result toward this conjecture was $\lambda_1\geq 3/4$ which was obtained by Sarnak [Sa]. Later he improved it to

$$\lambda_1\geq 21/25=0.84, \tag{1.6}$$

in his letter to Shahidi, which was published in [LRS2]. The bound (1.6) is the current best result for this case.

If φ is a Hecke eigen Maass cusp form attached to an eigenvalue λ, the automorphic L-function is defined by (1.2), and its gamma factor is given by

$$\Gamma(s,\varphi)=\Gamma_{\mathbb{C}}\left(s+\frac{ir}{2}\right)\Gamma_{\mathbb{C}}\left(s-\frac{ir}{2}\right),$$

where $\Gamma_{\mathbb{C}}(s)=(2\pi)^{-s}\Gamma(s)$ and $\lambda=1+r^2$. Here the conjecture (1.5) is again equivalent to $\mathrm{Re}(\pm ir)=0$. By taking $n=2$, $k=K_d$ and $\pi=\varphi$ in the theorem (1.4), we have $\mu_1=ir/2$ and $\mu_2=-ir/2$, and the crude estimate $|\mathrm{Re}(\pm ir)|<1$ holds.

In the next section we will review how the estimate of the first eigenvalue can be improved by use of the symmetric power L-functions, and will calculate some new results for three dimensional cases.

2 Results

As we saw in the previous section, the Selberg conjecture (1.1) is regarded as the Riemann Hypothesis for trivial zeros of the automorphic L-function $L(s,\varphi)$. Generally speaking, for improving a zero-free region for a Riemann type hypothesis, a possible useful idea is to take tensor products of a zeta functions, that is, to construct a zeta function having zeros at sum of zeros of the original zeta function. For our purpose the $(n-1)$-th symmetric power L-functions play the role:

(2.1)
$$L(s,\mathrm{Sym}^{n-1}(\varphi))=\prod_p\prod_{j=0}^{n-1}\left(1-\alpha_p^{n-1-j}\beta_p^jp^{-s}\right)^{-1}.$$

It is conjectured that the functions (2.1) have analytic continuations and functional equations. The meromorphic continuation and functional equation are known for these for $n\leq 10$ ([Sh]). Kim [K] and Kim-Shahidi [KS] established that the completed L-function supplied with the gamma factor is entire for $n=4$ and 5, except perhaps for poles at $s=0$ and 1. (The case $n=3$ was previously known by Shimura [S].) Moreover they show in these cases that there is an automorphic representation π_n on $GL_n(\mathbf{A}_k)$ whose L-function $L(s,\pi_n)$ is equal to $L(s,\mathrm{Sym}^{n-1}(\varphi))$. Here the number field k is typically given by

$$k=\begin{cases}\mathbb{Q} & (\dim M=2,\ M=\text{congruence type})\\ K_d & (\dim M=3,\ M=M_d)\end{cases}$$

In what follows we deal with these two cases, and will see how the first eigenvalue problem is investigated.

The above correspondence from φ to π_n is called the n-th symmetric power functorial lift from GL_2 to GL_n. This is a special but quite useful example of the general functoriality conjecture by Langlands [L]. We naturally set the following assumption on an integer $n\geq 2$ and a cuspidal automorphic representation π of $GL_2(\mathbf{A}_k)$ with a number field k.

Assumption 2.1 There is a cuspidal automorphic representation $\pi_n=\mathrm{Sym}^{n-1}(\pi)$ of $GL_n(\mathbf{A}_k)$ whose L-function is equal to $L(s,\mathrm{Sym}^{n-1}(\pi))$.

The results mentioned above assures that this is true for $n \leq 5$.

When Assumption 2.1 is true, the L-function $L(s, \pi_n) = L(s, \mathrm{Sym}^{n-1}(\pi))$ has an analytic continuation and the functional equation. The gamma factor is given by

$$(2.2) \quad \Gamma(s, \pi_n) = \begin{cases} \displaystyle\prod_{j=0}^{n-1} \Gamma_{\mathbb{R}}(s + (n-1-2j)ir) & (\dim M = 2) \\ \displaystyle\prod_{j=0}^{n-1} \Gamma_{\mathbb{C}}\left(s + \frac{n-1-2j}{2} ir\right) & (\dim M = 3). \end{cases}$$

Applying the theorem (1.4) to the Satake parameters

$$\mu_j = \begin{cases} (n-1-2j)ir & (\dim M = 2) \\ \dfrac{n-1-2j}{2} ir & (\dim M = 3) \end{cases}$$

with $j = 0$ gives

$$|\mathrm{Re}(ir)| < \begin{cases} \dfrac{1}{n-1} & (\dim M = 2) \\ \dfrac{1}{2(n-1)} & (\dim M = 3) \end{cases}$$

from which nontrivial estimates

$$(2.3) \quad \lambda_1 > \begin{cases} \dfrac{1}{4} - \left(\dfrac{1}{2(n-1)}\right)^2 & (\dim M = 2) \\ 1 - \left(\dfrac{1}{n-1}\right)^2 & (\dim M = 3) \end{cases}$$

follow. Thus we obviously see that Assumption 2.1 for all n implies the Selberg conjecture.

There is another remarkable approach to this problem from a different direction, which was created by Luo, Rudnick and Sarnak [LRS1] in 1995 by use of techniques in analytic number theory. They essentially improved Jacquet-Shalika's "critical strip" (1.4) as follows:

Theorem 2.2 ([LRS1, LRS2]) *When π is a cuspidal automorphic representation on $GL_n(\mathbf{A}_k)$, the estimate* (1.4) *is improved as*

$$|\mathrm{Re}(\mu_j)| \leq \frac{1}{2} - \frac{1}{n^2+1}.$$

They first proved the theorem for $k = \mathbb{Q}$ in [LRS1], which lead to the remarkable result $\lambda_1 \geq 21/100$ breaking the long lasting Selberg's record $3/16$. Then a generalization to any number field k is done in [LRS2], which they mention implies the same estimate for Shimura curves by our taking a totally real number field as k.

When we take an imaginary quadratic field as k, Theorem 2.2 also gives an improvement for three dimensional arithmetic manifolds such as Bianchi manifolds. We obtain the results by combining Theorem 2.2 with the above mentioned idea of tensor products of L-functions. The results are summarized as follows.

Theorem 2.3 *Let φ be a Hecke eigen Maass form corresponding to the first eigenvalue λ_1. If φ and an integer $n \geq 2$ satisfy Assumption 2.1, we have the following estimates:*

$$(2.4) \quad \lambda_1 > \begin{cases} \dfrac{1}{4} - \left(\dfrac{n+1}{2(n^2+1)}\right)^2 & (\dim M = 2) \\ 1 - \left(\dfrac{n+1}{n^2+1}\right)^2 & (\dim M = 3) \end{cases}.$$

The above mentioned results $\lambda_1 \geq 66/289$ by Kim and Shahidi and $\lambda_1 \geq 40/169$ by Kim for two dimensional cases agree to this result by our taking $n = 4$ and $n = 5$, respectively. By putting $n = 5$ in the three dimensional cases, we have a new estimate as follows.

Corollary 2.4 *For any Bianchi manofold $M = M_d$, the first eigenvalue λ_1 satisfies that*

$$\lambda_1 \geq \frac{160}{169} = 0.946\cdots.$$

Remark 2.5 When the class number of K is 1, more refined analysis as in the paper of Kim and Sarnak [KSa] is possible. If one would do the same as what they do, it should be shown that

$$|\mathrm{Re}(\mu_j)| \leq \frac{1}{2} - \frac{1}{\dfrac{n(n+1)}{2} + 1}$$

in place of Theorem 2.2 and consequently that

$$\lambda_1 \geq 975/1024 = 0.952\cdots.$$

3 Tensor Products of Zeta Functions

All the developments and the improvements toward the Selberg conjectures in the previous sections were obtained by the use of $(n-1)$-th symmetric power

L-functions. Their essential property is the fact that they have a zero at the $(n-1)$ times of zeros of the original automorphic L-functions. As we saw in the previous sections, the first eigenvalue problem is an analogue of the Riemann Hypothesis for trivial zeros of automorphic L-functions. It can be regarded as a general principle that tensor products of zeta functions are useful for solving problems in the type of the Riemann Hypothesis.

In the function field case, indeed, Deligne [D] heavily used the additive structure of zeros of congruence zeta functions in his proof of the Weil Conjecture. In that case the additivity of zeros of zeta functions are realized by the Kunneth's formula on étale cohomology. Hence it is natural to consider a tensor power for other zeta functions as well, such as the Riemann zeta function or Hasse zeta functions, although their underlying cohomology is not known. It should be essential to construct new zeta functions having zeros at sum of zeros of original zeta functions. The following theorem gives such functions.

Theorem 3.1 *Let p and q be prime numbers. Define the function $\zeta_{p,q}(s)$ as follows: for $p \neq q$*

$$\begin{aligned} &\zeta_{p,q}(s) \\ &:= \left(1-p^{-s}\right)^{\frac{1}{2}}\left(1-q^{-s}\right)^{\frac{1}{2}} \\ &\times \exp\left(\frac{1}{2i}\sum_{k=1}^{\infty}\frac{\cot\left(\pi k\frac{\log p}{\log q}\right)}{k}p^{-ks}\right. \\ &\qquad \left. +\frac{1}{2i}\sum_{n=1}^{\infty}\frac{\cot\left(\pi n\frac{\log q}{\log p}\right)}{n}q^{-ns}\right), \end{aligned}$$

and for $p = q$

$$\begin{aligned} &\zeta_{p,p}(s) \\ &:= \left(1-p^{-s}\right)^{1-\frac{is\log p}{2\pi}}\exp\left(-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns}\right). \end{aligned}$$

Then the function $\zeta_{p,q}(s)$ has the following properties:

(1) The sum over k and n in the definition of $\zeta_{p,q}(s)$ converges absolutely in $\mathrm{Re}(s) > 0$.

(2) The functions $\zeta_{p,q}(s)$ are meromorphic functions of order two on the entire plane.

(3) The zeros of $\zeta_{p,q}(s)$ are given by

$$s = 2\pi i\left(\frac{k}{\log p}+\frac{n}{\log q}\right)$$

with k and n nonnegative integers.

(4) The poles of $\zeta_{p,q}(s)$ are given by

$$s = 2\pi i\left(\frac{k}{\log p}+\frac{n}{\log q}\right)$$

with k and n negative integers.

Proof. The function $\zeta_{p,q}(s)$ agrees to the absolute tensor product $\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)$ except for a nonzero holomorphic factor, whose explicit form was given in our previous paper [KK2]. By the definition of the absolute tensor product, it has zeros at $\rho+\rho'$ with multiplicity $m(\rho,\rho')$, where ρ and ρ' denote zeros of $\zeta(s,\mathbf{F}_p)$ and $\zeta(s,\mathbf{F}_q)$, respectively, and the multiplicity is defined by

$$\begin{aligned} &m(\rho,\rho') \\ &= m_1(\rho)m_2(\rho')\times\begin{cases} 1 & \mathrm{Im}(\rho_j)\geq 0, \quad (j=1,2) \\ -1 & \mathrm{Im}(\rho_j) < 0, \quad (j=1,2) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the function $\zeta_{p,q}(s)$ has the desired property. ∎

The absolute tensor product was proposed by Kurokawa [Ku] in a purely analytic manner. In [KK2] we developed his theory to obtain the explicit Euler product of the double Hasse zeta function of finite fields. All the theory is based on our fundamental theory of multiple sine functions [KK1].

According to the definition of absolute tensor product we should count the combinations of zeros or poles, only when all of them are in the upper half plane or they are in the lower half plane. By this parity condition we obtain the results by establishing the following "signatured" double Poisson summation formula.

Theorem 3.2 ([KK2]) *Let $a, b > 0$. Assume a/b is generic and that the test function $H(t)$ satisfies*

$$\widetilde{H}(x) := \int_{-\infty}^{\infty}H(t)e^{itx}dt = O(\mu^x) \tag{3.1}$$

as $x\to\infty$ for some $0<\mu<1$, then we have

$$\begin{aligned} &\sum_{k,n>0}H\left(2\pi\left(\frac{k}{a}+\frac{n}{b}\right)\right) \\ &+ \frac{1}{2}\left(\sum_{k>0}H\left(2\pi\frac{k}{a}\right)+\sum_{n>0}H\left(2\pi\frac{n}{b}\right)\right) \\ &= -\frac{ia}{4\pi}\sum_{k>0}\cot\left(\pi\frac{ka}{b}\right)\widetilde{H}(ka) \\ &\quad -\frac{ib}{4\pi}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)\widetilde{H}(nb)-\frac{iab}{8\pi^2}\widetilde{H}'(0). \end{aligned}$$

Here the notion of genericity for real numbers a/b is defined in [KK2]. We deduce such a formula by taking a contour surrounding only the upper half of the critical strip and integrating the logarithmic derivative of the product of the zeta functions. Doing the same for the Riemann zeta function would lead us to the explicit form of the double Riemann zeta function expressed by the Euler product over pairs of prime numbers, after we establish the signatured double explicit formula. The details will be announced in our forthcoming paper.

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