Normalized Double Sine Functions

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Abstract

We express normalized double sine functions of integer periods (N_1, N_2) via the standard double sine function of period $(1, 1)$. As an application we give an Euler product expression using the di-logarithm for the double zeta function $\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$ for a prime number p and integers N_1, N_2.

1 Definitions and Results

Normalized multiple sine functions are generalizations of the usual sine function. We studied their basic properties in previous papers [KuKo] [KoKu1] [KoKu2] with some applications.

For $\omega_1, ..., \omega_r > 0$ and $x > 0$, the multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, x, (\omega_1, ..., \omega_r)) = \sum_{n_1,...,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}$$

in $\mathrm{Re}(s) > r$. This has the analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function, and it is holomorphic at $s = 0$. Then the normalized multiple gamma function is defined as

$$\Gamma_r(x, (\omega_1, ..., \omega_r)) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x, (\omega_1, ..., \omega_r)) \Big|_{s=0} \right).$$

This is a constant multiple of the multiple gamma function $\Gamma_r^B(x, (\omega_1, ..., \omega_r))$ of Barnes [B]:

$$\Gamma_r(x, (\omega_1, ..., \omega_r)) = \Gamma_r^B(x, (\omega_1, ..., \omega_r))/\rho_r(\omega_1, ..., \omega_r).$$

Now, the normalized multiple sine function is

$$S_r(x, (\omega_1, ..., \omega_r)) = \Gamma_r(x, (\omega_1, ..., \omega_r))^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x, (\omega_1, ..., \omega_r))^{(-1)^r}.$$

For example

$$\begin{aligned} S_1(x,\omega) &= \Gamma_1(x,\omega)^{-1}\Gamma_1(\omega-x,\omega)^{-1} \\ &= 2\sin(\pi x/\omega), \end{aligned}$$

since we have $\Gamma_1(x,\omega)=(2\pi)^{-1/2}\Gamma(x/\omega)\omega^{\frac{x}{\omega}-\frac{1}{2}}$ from $\zeta_1(s,x,\omega)=\omega^{-s}\zeta(s,x/\omega)$.

To simplify the notation we put $S_r(x)=S_r(x,(1,...,1))$, $\Gamma_r(x)=\Gamma_r(x,(1,...,1))$ and $\zeta_r(s,x)=\zeta_r(s,x,(1,...,1))$. Hence

$$S_r(x)=\Gamma_r(x)^{-1}\Gamma_r(r-x)^{(-1)^r}$$

and

$$\Gamma_r(x)=\exp\left(\frac{\partial}{\partial s}\zeta_r(s,x)\Big|_{s=0}\right).$$

Here we investigate normalized double sine functions, especially in the rational periods case: $S_2(x,(\omega_1,\omega_w))$ with $\omega_2/\omega_1\in\mathbb{Q}$. The following theorem expresses them in terms of $S_2(x)$:

Theorem 1.1 *Let N_1, N_2 be positive integers with the greatest common divisor N_0. Then we have*

$$S_2(x,(N_1,N_2))=\prod_{k_1=0}^{(N_2/N_0)-1}\prod_{k_2=0}^{(N_1/N_0)-1} S_2\left(\frac{x+N_1k_1+N_2k_2}{N_1N_2/N_0}\right). \tag{1.1}$$

As application of Theorem 1.1, we compute the absolute tensor product of the Hasse zeta functions of finite fields with p^{N_1} and p^{N_2} elements:

Theorem 1.2 *Let N_1, N_2 be positive integers with the greatest common divisor N_0. The absolute tensor product of the Hasse zeta functions for finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$ is given as follows:*

$$\begin{aligned} &\zeta(s,\mathbf{F}_{p^{N_1}})\otimes\zeta(s,\mathbf{F}_{p^{N_2}}) \\ &\quad=\exp\left(-\frac{1}{2\pi i}\frac{N_0^2}{N_1N_2}\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n^2}+\left(\frac{isN_0\log p}{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n}\right. \\ &\qquad\qquad\left.+\sum_{n=1}^{\infty}\frac{p^{-snN_1}}{n}f_1(n)+\sum_{n=1}^{\infty}\frac{p^{-snN_2}}{n}f_2(n)+Q_p(s)\right), \end{aligned}$$

where

$$f_1(n)=\begin{cases}(e^{2\pi inN_1/N_2}-1)^{-1} & (\frac{N_2}{N_0}\nmid n) \\ \frac{N_2-N_0}{2N_0} & (\frac{N_2}{N_0}|n)\end{cases}, \qquad f_2(n)=\begin{cases}(e^{2\pi inN_2/N_1}-1)^{-1} & (\frac{N_1}{N_0}\nmid n) \\ \frac{N_1-N_0}{2N_0} & (\frac{N_1}{N_0}|n)\end{cases}$$

and $Q_p(s)$ is a quadratic polynomial in s.

2 Proof of Theorem 1.1

It suffices to show when $N_0 = 1$, since the homogeneity [KuKo, Theorem 2.1(e)] of the multiple sine functions gives

$$S_2(x,(N_1,N_2)) = S_2\left(\frac{x}{N_0},\left(\frac{N_1}{N_0},\frac{N_2}{N_0}\right)\right).$$

In case $N_0 = 1$, the right hand side of (1.1) is calculated as follows:

$$\begin{aligned}
&\prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} S_2\left(\frac{x+N_1k_1+N_2k_2}{N_1N_2}\right)\\
=&\prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} \Gamma_2\left(\frac{x+N_1k_1+N_2k_2}{N_1N_2}\right)^{-1}\Gamma_2\left(2-\frac{x+N_1k_1+N_2k_2}{N_1N_2}\right)\\
=&\prod_{k_1=0}^{N_2-1}\prod_{k_2=0}^{N_1-1} \exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\left(-\zeta_2\left(s,\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)+\zeta_2\left(s,2-\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)\right)\right)\\
=&\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}\left(-\zeta_2\left(s,\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)+\zeta_2\left(s,2-\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)\right)\right)
\end{aligned} \tag{2.1}$$

The double sum is computed as follows:

$$\begin{aligned}
&\sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}\left(-\sum_{m_1,m_2\ge 0}\left(m_1+m_2+\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)^{-s}\right.\\
&\qquad\qquad\left.+\sum_{m_1,m_2\ge 0}\left(m_1+m_2+2-\tfrac{x+N_1k_1+N_2k_2}{N_1N_2}\right)^{-s}\right)\\
=&(N_1N_2)^s\sum_{k_1=0}^{N_2-1}\sum_{k_2=0}^{N_1-1}\left(-\sum_{m_1,m_2\ge 0}\Big((m_1N_2+k_1)N_1+(m_2N_1+k_2)N_2+x\Big)^{-s}\right.\\
&\left.+\sum_{m_1,m_2\ge 0}\Big((m_1N_2+N_2-k_1-1)N_1+(m_2N_1+N_1-k_2-1)N_2+N_1+N_2-x\Big)^{-s}\right)\\
=&(N_1N_2)^s\Big(-\zeta_2(s,x,(N_1,N_2))+\zeta_2(s,N_1+N_2-x,(N_1,N_2))\Big).
\end{aligned}$$

We previously obtained in the proof of [KuKo, Theorem 2.1(b)] that the function

$$-\zeta_2(s,x,(N_1,N_2))+\zeta_2(s,N_1+N_2-x,(N_1,N_2))$$

has zeros at even nonnegative integers s. In particular it vanishes at $s = 0$, thus (2.1) equals

$$\begin{aligned}
&\exp\left(\frac{\partial}{\partial s}\bigg|_{s=0}\Big(-\zeta_2(s,x,(N_1,N_2))+\zeta_2(s,N_1+N_2-x,(N_1,N_2))\Big)\right)\\
=\ &\Gamma_2(x,(N_1,N_2))^{-1}\Gamma_2(N_1+N_2-x,(N_1,N_2))\\
=\ &S_2(x,(N_1,N_2)).
\end{aligned}$$

This completes the proof of Theorem 1.1.∎

3 Application

In this section we compute the absolute tensor product of the Hasse zeta functions of finite fields $\mathbf{F}_{p^{N_1}}$ and $\mathbf{F}_{p^{N_2}}$. We first recall the definition of the absolute tensor product of meromorphic functions. Let Z_j $(j = 1, 2)$ be meromorphic functions of order μ_j. We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} \prod_{\rho\in\mathbb{C}}{}' P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}, \tag{3.1}$$

where $P_r(u) := (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r})$, m_j denotes the multiplicity function with $k_j := m_j(0)$, and Q_j is a polynomial with $\deg Q_j \le \mu_j$. Here the product over $\rho \in \mathbb{C}$ means $\lim\limits_{R\to\infty} \prod\limits_{0<|\rho|<R} P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}$. The absolute tensor product is defined by

$$(Z_1 \otimes Z_2)(s) := s^{k_1k_2} e^{Q(s)} \prod_{\rho_1,\rho_2\in\mathbb{C}}{}' P_{\mu_1+\mu_2}\left(\frac{s}{\rho_1+\rho_2}\right)^{m(\rho_1,\rho_2)}, \tag{3.2}$$

where $Q(s)$ is a polynomial with $\deg Q \le \mu_1 + \mu_2$ and

$$m(\rho_1,\rho_2) := m_1(\rho_1)m_2(\rho_2) \times \begin{cases} 1 & \text{if } \mathrm{Im}(\rho_1), \mathrm{Im}(\rho_2) \ge 0, \\ -1 & \text{if } \mathrm{Im}(\rho_1), \mathrm{Im}(\rho_2) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we do not give the precise definition of the polynomial $Q(s)$, since it is not necessary for our purpose.

In this section we will compute this absolute tensor product for the Hasse zeta functions for finite fields:

$$\begin{aligned}
Z_1(s) &= \zeta(s, \mathbf{F}_{p^{N_1}}) = (1 - p^{-N_1 s})^{-1}, \\
Z_2(s) &= \zeta(s, \mathbf{F}_{p^{N_2}}) = (1 - p^{-N_2 s})^{-1},
\end{aligned}$$

with p a prime number and N_1, N_2 positive integers.

The following theorem extends our previous results on $\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)$ with p, q primes in [KoKu2].

Proposition 3.1 *The absolute tensor product of the Hasse zeta functions for finite fields* $\mathbf{F}_{p^{N_1}}$ *and* $\mathbf{F}_{p^{N_2}}$ *is given as follows:*

$$\begin{aligned}\zeta(s,\mathbf{F}_{p^{N_1}})\otimes\zeta(s,\mathbf{F}_{p^{N_2}}) &= e^{Q(s)}S_2\left(\frac{isN_1N_2\log p}{2\pi},(N_1,N_2)\right)\\ &= e^{Q(s)}\prod_{k_1=0}^{(N_2/N_0)-1}\prod_{k_2=0}^{(N_1/N_0)-1}S_2\left(N_0\left(\frac{is\log p}{2\pi}+\frac{k_1}{N_2}+\frac{k_2}{N_1}\right)\right),\end{aligned}$$

where $N_0=(N_1,N_2)$ *and* $Q(s)$ *is a polynomial of degree at most two, which depends on* p*.*

Proof. The second equality is seen from Theorem 1.1. In what follows we prove the first one. The Hadamard product (3.1) for the Hasse zeta function is given for $j=1,2$ by

$$\zeta(s,\mathbf{F}_{p^{N_j}})=s^{-1}e^{\tilde{Q}_{N_j}(s)}\prod_{n=-\infty}^{\infty}{}' P_1\left(\frac{s}{\frac{2\pi i}{N_j\log p}n}\right)^{-1}$$

with $\tilde{Q}_{p,j}(s)$ a linear polynomial depending on N_j. Thus by the definition (3.2) of the absolute tensor product,

$$\zeta(s,\mathbf{F}_{p^{N_1}})\otimes\zeta(s,\mathbf{F}_{p^{N_2}})=se^{\tilde{Q}_{N_1,N_2}(s)}\prod_{k,n\in\mathbb{Z}}{}' P_2\left(\frac{s}{\frac{2\pi i}{N_1\log p}k+\frac{2\pi i}{N_2\log p}n}\right)^{m_{k,n}},$$

where $\tilde{Q}_{N_1,N_2}(s)$ is a polynomial of degree at most two and

$$m_{k,n}:=m\left(\tfrac{2\pi i}{N_1\log p}k,\tfrac{2\pi i}{N_2\log p}n\right)=\begin{cases}1 & \text{if } k,n\geq 0\\ -1 & \text{if } k,n<0\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\zeta(s,\mathbf{F}_{p^{N_1}})\otimes\zeta(s,\mathbf{F}_{p^{N_2}})=e^{\tilde{Q}_{N_1,N_2}(s)}s\frac{\prod_{k,n=0}^{\infty}{}' P_2\left(\frac{s}{\frac{2\pi i}{N_1\log p}k+\frac{2\pi i}{N_2\log p}n}\right)}{\prod_{k,n=1}^{\infty}P_2\left(-\frac{s}{\frac{2\pi i}{N_1\log p}k+\frac{2\pi i}{N_2\log p}n}\right)}.$$

We appeal to the $r = 2$ case of the formula [KuKo, Proposition 2.4]:

$$S_2(z, (\omega_1, \omega_2)) = e^{Q_{\underline{\omega}}(z)} z \frac{\prod'^{\infty}_{k,n=0} P_2\left(-\frac{z}{\omega_1 k + \omega_2 n}\right)}{\prod^{\infty}_{k,n=1} P_2\left(\frac{z}{\omega_1 k + \omega_2 n}\right)}$$

where $Q_{\underline{\omega}}(z)$ a polynomial with $\deg Q_{\underline{\omega}} \leq 2$. The proof is complete.∎

Taking the following exponential expression of the normalized double sine functions into account, we see that the absolute tensor product in Theorem 1.2 has an "Euler product" expression for $\mathrm{Re}(s) > 0$.

Proposition 3.2 *The following expression holds for* $\mathrm{Im} z > 0$.

$$S_2(z) = \exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z}) + (1 - z)\log(1 - e^{2\pi i z}) + Q(z)\right),$$

where $Q(z) = \frac{\pi i}{2} z^2 - \pi i z + \frac{5\pi i}{12}$.

Proof. We recall the formulas of the double sine functions:

$$S_2(z) = \mathcal{S}_2(z)^{-1}\mathcal{S}_1(z), \qquad \text{([KuKo, Example 3.6])} \tag{3.3}$$

where $\mathcal{S}_r(z)$ $(r = 1, 2)$ are the primitive multiple sine functions [KuKo]. We have by definition

$$\mathcal{S}_1(z) = 2\sin \pi z = \exp\left(-\pi i z + \frac{\pi i}{2} + \log(1 - e^{2\pi i z})\right)$$

and the expression [KuKo, Theorem 2.8 (2.12)]:

$$\mathcal{S}_2(z) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z}) + z\log(1 - e^{2\pi i z}) - \frac{\pi i}{2}z^2 - \frac{\zeta(2)}{2\pi i}\right)$$

for $\mathrm{Im}(z) > 0$. Thus (3.3) equals

$$\exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z}) + (1 - z)\log(1 - e^{2\pi i z}) + \frac{\pi i}{2}z^2 - \pi i z + \frac{5\pi i}{12}\right).$$ ∎

Lemma 3.3 *Assume* $r \in \mathbb{C}$ *satisfies that* $r^N = 1$. *Then we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} n r^n = \begin{cases} (r-1)^{-1} & (r \neq 1) \\ \frac{N-1}{2} & (r = 1). \end{cases}$$

Proof. The $r=1$ case is well-known. Differentiating the formula $\sum_{n=0}^{N-1} r^n = (1-r^N)/(1-r)$ in case $r \neq 1$ leads to the result. ∎

Proof of Theorem 1.2: By the above propositions all we should compute is the following product:

$$\prod_{k_1=0}^{(N_2/N_0)-1} \prod_{k_2=0}^{(N_1/N_0)-1} S_2\left(N_0\left(\frac{is\log p}{2\pi} + \frac{k_1}{N_2} + \frac{k_2}{N_1}\right)\right)$$

$$= \exp\left(\sum_{k_1=0}^{(N_2/N_0)-1} \sum_{k_2=0}^{(N_1/N_0)-1} \left(-\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{-N_0 s\log p + 2\pi i(\frac{N_0k_1}{N_2}+\frac{N_0k_2}{N_1})}\right)\right.\right. \tag{3.4}$$

$$+\left(1-\frac{isN_0\log p}{2\pi} - \frac{N_0k_1}{N_2} - \frac{N_0k_2}{N_1}\right)\log\left(1-e^{-sN_0\log p + 2\pi i(\frac{N_0k_1}{N_2}+\frac{N_0k_2}{N_1})}\right) \tag{3.5}$$

$$\left.\left. +Q\left(\frac{isN_0\log p}{2\pi} + \frac{N_0k_1}{N_2} + \frac{N_0k_2}{N_1}\right)\right)\right). \tag{3.6}$$

Put $N_1 = N_0N_1'$ and $N_2 = N_0N_2'$. First the double sum in (3.4) is computed as follows:

$$
\begin{aligned}
& -\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{-N_0 s\log p + 2\pi i(\frac{k_1}{N_2'}+\frac{k_2}{N_1'})}\right) \\
= & -\frac{1}{2\pi i}\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\sum_{n=1}^{\infty}\frac{\left(p^{-sN_0}e^{2\pi i(\frac{k_1}{N_2'}+\frac{k_2}{N_1'})}\right)^n}{n^2} \\
= & -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{p^{-sN_0n}}{n^2}\left(\sum_{k_1=0}^{N_2'-1} e^{2\pi i\frac{nk_1}{N_2'}}\right)\left(\sum_{k_2=0}^{N_1'-1} e^{2\pi i\frac{nk_2}{N_1'}}\right) \\
= & \frac{1}{2\pi i} - \sum_{N_1'N_2'|n}\frac{p^{-sN_0n}}{n^2}N_1'N_2' \\
= & \frac{1}{2\pi i} - \frac{1}{N_1'N_2'}\sum_{n=1}^{\infty}\frac{p^{-sN_0N_1'N_2'n}}{n^2} \\
= & \frac{1}{2\pi i} - \frac{N_0^2}{N_1N_2}\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n^2}.
\end{aligned}
$$

Next we compute the double sum of (3.5) over k_1, k_2:

$$\begin{aligned}
&\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\left(1-\frac{isN_0\log p}{2\pi}-\frac{k_1}{N_2'}-\frac{k_2}{N_1'}\right)\log\left(1-e^{-sN_0\log p+2\pi i(\frac{k_1}{N_2'}+\frac{k_2}{N_1'})}\right)\\
=\ & -\left(1-\frac{isN_0\log p}{2\pi}\right)\sum_{n=1}^{\infty}\frac{p^{-sN_0n}}{n}\sum_{k_1=0}^{N_2'-1}e^{2\pi i\frac{nk_1}{N_2'}}\sum_{k_2=0}^{N_1'-1}e^{2\pi i\frac{nk_2}{N_1'}}\\
&+\sum_{n=1}^{\infty}\frac{p^{-sN_0n}}{n}\left(\sum_{k_1=0}^{N_2'-1}\sum_{k_2=0}^{N_1'-1}\left(\frac{k_1}{N_2'}+\frac{k_2}{N_1'}\right)e^{2\pi i(\frac{k_1}{N_2'}+\frac{k_2}{N_1'})n}\right)\\
=\ & \left(\frac{isN_0\log p}{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{p^{-sN_0nN_1'N_2'}}{n}\\
&+\sum_{n=1}^{\infty}\frac{p^{-sN_0n}}{n}\left(\sum_{k_1=0}^{N_2'-1}\frac{k_1}{N_2'}e^{2\pi i\frac{k_1n}{N_2'}}\sum_{k_2=0}^{N_1'-1}e^{2\pi i\frac{k_2n}{N_1'}}+\sum_{k_2=0}^{N_1'-1}\frac{k_2}{N_1'}e^{2\pi i\frac{k_2n}{N_1'}}\sum_{k_1=0}^{N_2'-1}e^{2\pi i\frac{k_1n}{N_2'}}\right)\\
=\ & \left(\frac{isN_0\log p}{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{p^{-sN_0nN_1'N_2'}}{n}\\
&+\sum_{n=1}^{\infty}\frac{p^{-snN_0N_1'}}{nN_1'}\sum_{k_1=0}^{N_2'-1}\frac{k_1}{N_2'}e^{2\pi i\frac{k_1nN_1'}{N_2'}}N_1'+\sum_{n=1}^{\infty}\frac{p^{-snN_0N_2'}}{nN_2'}\sum_{k_2=0}^{N_1'-1}\frac{k_2}{N_1'}e^{2\pi i\frac{k_2nN_2'}{N_1'}}N_2'\\
=\ & \left(\frac{isN_0\log p}{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{p^{-sN_0nN_1'N_2'}}{n}\\
&+\sum_{n=1}^{\infty}\frac{p^{-snN_0N_1'}}{nN_2'}\sum_{k_1=0}^{N_2'-1}k_1e^{2\pi i\frac{k_1nN_1'}{N_2'}}+\sum_{n=1}^{\infty}\frac{p^{-snN_0N_2'}}{nN_1'}\sum_{k_2=0}^{N_1'-1}k_2e^{2\pi i\frac{k_2nN_2'}{N_1'}}\\
=\ & \left(\frac{isN_0\log p}{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n}+\sum_{n=1}^{\infty}\frac{p^{-snN_1}}{n}f_1(n)+\sum_{n=1}^{\infty}\frac{p^{-snN_2}}{n}f_2(n)
\end{aligned}$$

by Lemma 3.3. ∎

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